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ELEMENTARY GEOMETRY

CONGRUENT FIGURES

BY

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PROFESSOR OF PURE MATHEMATICS IN UNIVERSITY COLLEGE LONDON

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EDITORS' PREFACE.

NOTWITHSTANDING the large number of scientific works which have been published within the last few years, it is very generally acknowledged by those who are practically engaged in Education, whether as Teachers or as Examiners, that there is still a want of Books adapted for school purposes upon several important branches of Science. The present SERIES will aim at supplying this deficiency. The works comprised in the SERIES will all be composed with special reference to their use in school-teaching; but, at the same time, particular attention will be given to making the information contained in them trustworthy and accurate, and to presenting it in such a way that it may serve as a basis for more advanced study.

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fully as possible the nature of the methods of inquiry and reasoning by which these results have been obtained. Consequently, although the treatment of each subject will be strictly elementary, the fundamental facts will be stated and discussed with the fulness needed to place their scientific significance in a clear light, and to show the relation in which they stand to the general conclusions of Science.

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G. C. F.,
P. M.

AUTHOR'S PREFACE.

GEOMETRY is the science of Space. In its fullest meaning, it embodies the knowledge and the investigation of properties of space, however simple and familiar or however intricate they may be, and by whatever process this knowledge may have been acquired.

To it belongs the knowledge gained by a systematic study, which, begun by Egyptian priests in ages long gone by and continued through many centuries, is at the present day carried on more vigorously than ever; to it also belongs the knowledge unconsciously obtained by us while living and moving in space. As everything we do or perceive is in space, a large amount of our experience must of necessity relate to it. A carpenter or a mechanician may, during the pursuit of his calling, obtain a large stock of real geometrical knowledge without knowing anything about a proof, or without even knowing that his knowledge has anything to do with Geometry.

Experience developes principally the faculty of

realising to the mind arrangements of things in space, and the art of dealing practically with them.

The 'science' of Geometry, on the other hand, requires a systematic analysis of the properties of space, beginning with a study of the simplest and going on to that of the most complicated figures,—the simplest figures being not necessarily those with which we meet most frequently in everyday experience. Further, each new property stated has to be shown by a rigid proof actually to contain a general geometrical truth.

The two modes of acquiring geometrical knowledge thus indicated, different as they are in their nature, nevertheless react continuously upon each other.

The 'science' sprung originally from a desire to systematise previous, and to guide further, experience. The axioms which form its very basis, are obtained by experience, and its study remains barren without constant recourse to the inspection of solids or their substitutes, geometrical figures. Reciprocally, the faculty above referred to of comprehending figures in space and forming clear mental pictures of them, and the art of making practical use of this faculty, are much assisted by the methodical study of Geometry.

In teaching, both sides ought to be kept in view. The study of the science of Geometry can only be carried on satisfactorily if the student possesses a sufficient amount of knowledge gained by experience.

Where this is wanting, or where the connection between his experience and the science is not brought home to him, the student will be unable to make any progress: in most cases, I believe, not because he is unable to understand exact reasoning, but simply because he cannot connect the subject reasoned about with any concrete notions he has already acquired. This is, in my opinion, the reason why so many boys fail to understand Euclid, and fall back in desperation on the expedient of learning propositions by heart. This lack of concrete geometrical notions could not exist if all children, either in a *Kindergarten* or in their play at home, were early made familiar with the simplest forms and their most obvious mutual relations.

In order, however, to refer constantly to the concrete, geometrical drawing ought to be combined systematically with the teaching of Geometry. This is scarcely possible in connection with Euclid; and a student who wishes to make practical use of Geometry has to make a separate study of geometrical drawing, whether or not he has already studied and mastered Euclid. Geometrical drawing belongs, in fact, to a branch of Geometry of which Euclid knew nothing, and where Euclid's propositions are of little use. This branch, itself the outcome of an attempt made nearly a century ago to systematise the art of drawing as developed during past ages by handicraftsmen, is now

known as projective or modern Geometry, and has assumed such dimensions as almost to include the whole science.

I have tried to put the subject in such a manner that the student may thoroughly master the first elements of Geometry, and that he may realise the geometrical contents of the propositions as properties of space through actually seeing their truth by the mental or physical inspection of figures, instead of being convinced of their truth by a long process of logical reasoning. To attain this object it is necessary in the proof of each proposition to go back to first principles as far as this is possible; and this may be done to a very great extent by introducing a notion which is extremely simple, though of great generality. This notion is that of the correspondence of points or lines in two figures which are identically equal or 'congruent,' corresponding points or lines being defined as points or lines which coincide when the figures themselves are made to coincide. This notion is afterwards easily extended to similar figures, and more generally to figures which are projective.

Just as in higher Geometry the investigation of projective figures is simplified by placing them in perspective positions, so I bring congruent figures not into a position of coincidence, but into a position of *symmetry, which is* the perspective position of con-

gruent figures. This can be done in two ways reciprocal to each other, and thus the principle of duality is illustrated at an early period by an important example. This principle I have introduced from the beginning in pointing out the reciprocity between angles and segments of lines, that is, of straight lines of finite length, and I have dwelt on its importance throughout the book. For this reason I have introduced the figures reciprocal to loci by considering what I call 'sets of lines,' i.e. the aggregate of all lines which satisfy a given condition, and I have considered the circle first as a locus of points and then as the envelope of a set of lines.

Another innovation consists in the early introduction of the notion of 'sense' in a line or an angle, corresponding to the signs (+ and −) in algebra.

A number of exercises has been added to each chapter. Many of these refer to geometrical drawing, and include instructions about the use of instruments. Besides the straight-edge and a pair of compasses, the student is supposed to possess the two usual kinds of set squares. Preference is given to those constructions which are performed by aid of straight-edge and set squares, the use of the compasses being avoided as much as possible.

Some of the exercises refer to the logic of the proofs, and especially to the logical connection be-

tween propositions with reference to a 'Digression on Logic' which has been inserted between Chapters II. and III.

To give a preliminary notion of the order in which the subject has been arranged, I add a short statement of the contents of the several chapters.

The first chapter contains the fundamental notions of Geometry, viz. those of point, curve, surface, solid, and space, and the first three axioms. Next the straight line and the plane are considered, and two more axioms are given. On this follow figures, consisting of either two points or two lines which intersect. The two points give rise to a finite straight line called a segment, the two lines to an angle. A separate chapter is devoted to the bisections of such segments and angles. Special attention has been paid to fixing the 'sense' of a segment or an angle, and some of the simplest of Möbius' equations have been stated. Next, parallel lines are investigated, and the sixth and last axiom is introduced. The theorems about angles in polygons, which are immediate consequences of the theory of parallels, follow, and give first theorems about figures consisting of more lines and points. With this the foundation of Geometry has been laid. The fundamental notions, the axioms, and the simplest figures are given, and the study of more complicated figures can be undertaken.

The rest of the volume is devoted to the investi-

gation of the theory of figures which are equal in all respects, and which I call congruent, and to consequences following from this theory. It coincides in its contents with the first four books of Euclid, with the exception of the theorems relating to areas. The treatment, however, is very different. In the eighth chapter congruent figures, which are defined as figures that can be made to coincide, are considered quite generally. It is pointed out that to every point or line in one 'corresponds' a point or a line in the other, and generally that to every part in the one there is a 'corresponding' part in the other. Then congruent figures are brought into special positions. They are first made to coincide, and then in their common plane either a line or a point is fixed. The one figure is now with its plane turned about the fixed line till it falls again into the plane, or it is turned in its plane about the fixed point. The figures are then said to be symmetrical either with regard to an axis or with regard to a centre of symmetry. Symmetrical figures are thus congruent figures in special positions; they are the most special cases of projective figures in perspective position. Some of the most important properties due to this position are next stated. The proofs are so simple that they are in most cases only indicated, or even altogether omitted. Every reader, especially every teacher, will easily complete them if

he has grasped the meaning of *correspondence in symmetry.*

After this general investigation, which each teacher can give to his pupils all at once, or only gradually as wanted, the study of special figures is begun. First single triangles and quadrilaterals are considered, the beginning being made with the symmetrical triangle, which of course is Euclid's isosceles triangle. From the fact that it has an axis of symmetry all its properties follow at once. The theorem that the greater side in a triangle. is opposite the greater angle, and others connected with it, follow with equal ease.

Of symmetrical quadrilaterals there are three, one with a centre, the parallelogram, and two with an axis. Of these, one has a diagonal as axis of symmetry, and this I call a *kite*, adopting the name given to it by Prof. Sylvester; the other is the symmetrical or isosceles trapezium. Their properties all follow from their symmetry.

After this I have inserted a chapter on congruent triangles. Of their theory, however, little use is made in the sequel. Then follows a chapter on 'loci of points' and 'sets of lines,' as explained before. The rest of the book is devoted to the circle, which is first treated as a locus and then as the envelope of a set of lines. The circle being the embodiment of symmetry, I need not dwell on the ease with

which its properties follow from considerations based on symmetry.

I have undertaken this book from the desire to prepare students from the very first for those modern methods of which the method of projection and the principle of duality are the most fundamental. The advantages of the method adopted will, however, be fully appreciated only in their continuation in the second volume, which will treat of areas in connection with what Möbius calls 'equal figures' and of similar figures. These figures in their perspective positions follow from the two kinds of symmetrical figures, by dropping one measurement in each case. From axial symmetry we obtain 'equal' or skew-symmetrical figures by drawing the lines joining corresponding points not perpendicular to the axis, whilst similar figures similarly situated are obtained from central symmetry by dropping the condition that corresponding points shall be equidistant from the centre.

O. HENRICI.

UNIVERSITY COLLEGE, LONDON:
December, 1878.

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PLANE GEOMETRY.

PART I.

CONGRUENT FIGURES.

CHAPTER I.

FUNDAMENTAL NOTIONS.

§ 1. IF we set ourselves the problem of investigating the properties of things which we observe, we are very soon led to subdivide it into a number of different problems, according to the nature of the properties on which we fix our attention. This leads to the subdivision of natural science into various branches. Of these the science of mathematics is the simplest, and, consequently, it has reached the highest stage of development. It treats of the most general properties of things. The first notion which anything suggests to us is that of its own existence. If in considering anything we take account only of the fact of its existence, we obtain the notion of a *unit*; and by *considering* the existence of things in a group we get

the notion of a group of units—that is, of a *number*. This leads to the sciences of *Arithmetic* and *Algebra*, with which, however, we are not concerned at present. They alone constitute, strictly speaking, *pure* mathematics. As we gradually take more and more properties into consideration we are led in succession to branches of science of greater and greater complexity.

Thus in *Geometry* we consider the shape, size, position, and motion of things. The introduction of the notion of Time leads us from Geometry to *Kinematics*. From this science we are brought to that of *Kinetics*, or Dynamics, by adding the notions of Matter and Force. Here matter is considered only as having inertia and as being acted upon by force. Other properties of matter which manifest themselves in the phenomena called heat, light, electricity, and so on, are investigated in *Physics*, whilst in *Chemistry* the differences in kind of matter are studied. The much more complex conditions and changes of life as observed in plants and animals constitute the subject-matter of *Biology*.

§ 2. **Geometry.**—Geometry, which is the branch of science with which we are at present concerned, treats of some of the properties which are *common* to all such things as are cognisable by the senses of touch and sight.

§ 3. **Shape.**—By these senses we are led to perceive that bodies differ widely in colour, weight, temperature, and in many other properties, all of which depend more or less on the material out of which the bodies are formed. Other properties, however, are independent of the material—as, for example, the *shape*

of a body. Thus two spheres or globes have the same shape: though the one may be made of iron, the other of wood or marble, still they are globes—that is, things having a peculiar, definite *shape.*

§ 4. **Size.**—Again, if we take two solids of the same shape—say, two globes—these need not be equal. The one may be small, the other large. We express this by saying that the two bodies, though of the same shape, are of different *sizes.* On the other hand, two globes, though of the same size, may be of different material, so that size, like shape, does not depend upon material.

§ 5. **Position.**—But even if we have two solids of the same size, of the same shape, and similar in all other respects, so that, considering them each by itself, it is impossible to tell which is which, they are still not the same. They are distinguished from each other by occupying different *positions* in space.

Two material bodies cannot occupy the same space. We are thus led to recognise a third property common to all bodies: every body has *position.*

§ 6. **Motion.**—This brings us to the last property which we have to consider in geometry. *A body may change its position, and may be moved about in space*—that is, it may assume different positions at different times.

§ 7. We have thus obtained, by appealing to universal experience, four distinct properties which are common to all bodies, but are independent of their material. These are *shape, size, position,* and *capability of being moved*; and they are the only properties with which geometry is concerned.

Whatever other properties a body may possess, we leave them out of consideration, and treat them practically as though they did not exist.

§ 8. **Space.**—The four geometrical properties mentioned above all refer to *space.* Space itself is a fundamental conception which it is impossible to define or even to describe. Everything that we observe is in space, and space extends around us in all directions.

§ 9. We may now define geometry as the science which treats of the *properties of space.*

Of space itself we derive the first and fundamental properties from experience. These fundamental properties are laid down in propositions, called *axioms.* Before these can be stated we must develope the notions of the above-mentioned properties of bodies somewhat more fully.

§ 10. **Solids.**—We have seen that the *geometrical* properties of a body do not depend on the matter composing it, and therefore we must consider them as remaining unchanged, even if we could conceive that the matter ceased to exist.

The embodiment of these residual properties constitutes the geometrical notion of a solid.

When, therefore, the word *solid* is used in geometry, it is to be taken in this sense. It will be seen that such a geometrical solid can exist only as a mental conception ; it can have no material existence, but the idea of it is obtained by intellectually abstracting the non-geometrical properties from material bodies.

A *solid* may then be defined, or rather described, as a *part of space bounded on all sides.*

§ 11. **Surfaces.**—That which bounds a solid and

separates it from other parts of space is called its *surface.* If, for example, we consider a tumbler with water, then the water, if at rest, will occupy a definite part of space and have a definite shape. The boundary of this part of space separates the water at the bottom and the sides from the glass of the tumbler, and on the top from the air in the room. The boundary between the water and glass forms part neither of the water nor of the glass, but separates the one from the other, so that where the one ends the other begins. This is expressed by saying *a surface has no thickness.*

A surface is in space, but is not a part of it in the same sense as a solid is. It has, however, size, shape, and position, and may be moved.

§ 12. **Curves.**—A surface may consist of different parts. Thus, one part of the surface of the water in the tumbler consists of the boundary between water and glass, while another part separates the water from the air. Each of these parts is bounded where it meets the other part.

The boundary of a surface or of part of a surface is called a *line* or a *curve.*

The edges of solids—the edges of a square box, for instance—are lines. Curves and lines, as well as surfaces, have shape, size, and position, and may be moved with the surface or solid on which they lie.

§ 13. **Points.**—A line or curve may itself be bounded.

The edges of a square box meet, and are bounded, at the corners of the box.

The boundaries of a line or curve are called points.

A point has neither size nor shape ; but it has position, and may be moved ; for we may move the solid on which it lies.

A point, then, as long as we do not move it, has only one property : it has *position* only, and marks a place in space.

§ 14. We have thus obtained the fundamental notions with which we have to deal in geometry, viz. the notions of *space*, of *solids*, *surfaces*, *curves*, and *points*.

The following considerations will greatly assist in getting a clearer conception of them.

§ 15. As it is the solid of which the notion is given directly by experience, we make it again our starting-point. We suppose that a solid gets reduced in size, and becomes smaller and smaller without limit, till at last it loses all size and, with its size, its shape. This gives us the notion of a *point.* The only properties which it retains are *position* and *capability oj being moved.* Such a point does not exist in the material world. It is, in fact, an *abstraction.*

A point has no extension, and this must be carefully remembered. We may conceive points everywhere in space ; their number is unlimited. But if we bring different points together into the same position they will never give us anything but a point; we never obtain any extension. We cannot, therefore, say that space is made up of points, although space *contains* an unlimited number of them.

§ 16. **Path of Moving Point.**—But a point may be moved, and then it will describe a path. *This path of a moving point is a curve.* Again, a curve

contains an unlimited number of points. We may take any number of points on a curve, but, however near we take them, there will be room on the curve for other points between them. When two points come together they coincide and form one point only.

The notion of a line may be obtained directly by considering a wire bent into any shape and abstracting all thickness from it.

§ 17. **Sense of Motion.**—Suppose now we take two points on a curve—we may distinguish them by calling[1] one the point A and the other the point B—then the moving point by which we suppose the curve to be described may either come first to A and afterwards to B, or may first come to B and then to A. The curve may, therefore, be described by the point in two different ways. This is expressed by saying that the point moves along the curve either in one or in the opposite *sense.* Also the curve itself is said to have a sense, or to be of one or the other sense, according as it is considered as being described by a point moving in the one or in the other sense. For instance, a person may go from London to Brighton, or by the same road from Brighton to London. Going from north to south he would traverse the distance in one sense: going from south to north he would traverse it in the opposite sense. Similarly, the sense of a motion from left to right is opposite to that of a motion from right to left. This difference

FIG. 1.

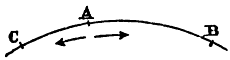
[1] A point will in future be denoted by a capital letter, like A, B, C, or M, P, &c., and, as a rule, capital letters will always *denote points*, and points only.

in sense is often distinguished by calling the one sense positive and the other negative. We may call either of them positive, and then the other is fixed as negative. We shall, as a rule, consider the motion from left to right as positive.

The sense is generally denoted by an arrow-head.

§ 18. If we consider a point A on a curve, we may move another point from it along the curve either in the one sense to B, or in the opposite sense to C. These two points are said to be on *opposite sides* of A. A lies between C and B, and *separates* them. A point *in a curve* has, therefore, *two sides.*

FIG. 2.



§ 19. **Dimension of a Curve.**—The different points in a curve follow each other in such a manner that we can pass in only two different ways from a point A to other points; we must move either in the one or in the opposite sense. This fact is expressed by saying that a curve is of *one dimension.* It has *length.*

§ 20. **Space is Unlimited.**—A point, in describing a curve, may either return to its original position—and then the curve is said to be *closed*—or the point may move to a greater and greater distance from its first position; and to this process we cannot, from experience, conceive any limit. Curves, then, may be *closed*, or they may *extend to an indefinite distance.*

This shows also *that space itself must be considered to be of an indefinite extension.* We cannot, indeed, conceive a limit to space, but neither can we comprehend anything infinite or of indefinite extension. Our experience is limited and leaves us here in the dark.

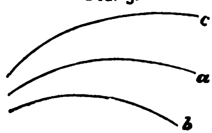
§ 21. **Arc of a Curve.**—Any part of a curve bounded by two points, A and B, may be called an *arc*, or a *segment* of the curve, and may be denoted by AB or by one small letter, say, a.[1]

§ 22. An arc of a curve may be moved without changing its shape or size. It will thus describe a path; this path may in a few special cases be the original curve again. This, however, is possible only if the moving curve be a part of a straight line, of a circle, or of a helix (the thread of a screw is a helix), for such a curve may slide along itself.

§ 23. **Path of a Moving Curve.**—In all other cases the path of a moving curve is not a curve, but something different which is called a surface. *A surface is the path of a moving curve.*

§ 24. If we consider any two positions successively occupied by the describing curve, and denote these positions by a and b, then the moving curve may either come first to a and afterwards to b, or it may move in the opposite *sense* by arriving first at b and then at a. Hence if we take any position, a (fig. 3), we may pass to another position of our curve by moving from a either in one sense to a position b, or by moving in the opposite sense to a position c. This shows that a curve *on a surface* has *two sides*, and further that curves generating a surface follow each other like the points on a curve.

FIG. 3.



[1] A curve will in future be denoted by a small letter—a, b, c, or p, q, r, &c.—and small letters will, as a rule, be used to denote curves or parts of curves only.

§ 25. **Dimensions of a Surface.**—From one position of the moving curve we may pass to other positions by moving either in one or in the opposite sense. Hence, if we consider the surface as generated by the motion of a curve, we may say the surface is of *one dimension* with regard to the describing curve, which in this case is considered as an *element.* The curve is here, in fact, considered not as generated by the motion of a point, but given as a whole. But if we consider the curve as containing points, or as being described by a moving point, it is itself of one dimension, and then the surface is said to be of *two dimensions* with regard to the points as elements.

§ 26. **Spreads.**—The word 'dimension' has here been used in a meaning different from that generally attached to it. In order to avoid confusion, which might arise by stating of a surface at one time that it has one, at another time that it has two dimensions, a different nomenclature has been proposed.

We have seen that a curve contains an infinite number of points, which may be considered as being spread along the curve. Similarly points and curves are spread over the surface, and also throughout space. A curve, or a surface, may therefore be called a *spread.* A surface is a spread with regard to points, as well as with regard to curves as elements.

There are thus different kinds of *spreads.* A curve, or any other spread on which the elements follow like the points in a line, is said to be one-wayed, or to be a one-way spread.

Hence a curve is a *one-way spread, with points as elements.* A surface, as generated by a moving curve,

is a *one-way spread, with the curves as elements* ; but a surface is a *two-way spread, with points as elements.*

§ 27. **Dimension.**—The word 'dimension' is retained to denote the number of ways of a spread, considering the points as elements, and then we may say a *curve is of one dimension, called length* ; and *a surface is of two dimensions, called length and breadth.*

§ 28. The use of the word 'spread' will be understood more fully if we apply it to non-geometrical notions.

Time extends from an indefinite past to an indefinite future ; it contains an unlimited number of moments. From any given moment of time we may, mentally, go in two different ways to other moments, either to an earlier or a later moment. The moments of time, therefore, follow each other like points in a curve ; and thus we may say that time is a *spread of moments.* As we can pass from one moment to another moment only in one or in the opposite way, it is a one-way spread. Hence *time is a one-way spread, with moments as elements.*

In a similar manner the temperature of a body may be changed. This, again, can be done only in one or in the opposite way ; we may either raise or lower it. The change of temperature takes place, therefore, either in one *sense* or in the opposite *sense.*

Temperature, consequently, is a *one-way spread,* with degrees of temperature as elements.

In a similar manner the *weight* of a body is a *one-way spread,* and so on.

As another interesting example let us consider a

musical tone of a certain pitch. We may, without changing its pitch, vary it in loudness or intensity; and, as this may be done in one or in the opposite sense, we have again a one-way spread of variations of intensity.

But we may also change its pitch in one or in the opposite sense, making the pitch higher or lower. Hence we may change a musical tone in two ways, each giving rise to a spread, which is very different from the other. The one is the way along which intensity changes, the other is the way along which pitch changes. We therefore say that a musical tone allows of variations which form a two-way spread, with different degrees of intensity and of pitch as elements.

The number of examples might be multiplied by considering light and colour; but the above will be sufficient to illustrate the meaning of the word 'spread.' Only one point has still to be considered.

§ 29. **Continuity.**—In all the above examples the variation of the elements was of a peculiar kind. A point, in changing its position on a curve, passes, in moving from one position to another, through all intermediate positions. It does not move by jumps. This is expressed by saying that the point changes its position *continuously*, that the curve is *continuous*.

If we take, instead of the *curve*, a row of solids—say, peas, laid in a row—we have not a continuous change in going from one to the other.

In order that an aggregate of elements, whatever these may be, may be called a spread, it is necessary that these elements follow continuously.

Thus the natural numbers 1, 2, 3, 4 . . . do not form a spread, as there is not a continuous change between them.

Aristotle indicated this difference by saying that a row of elements is *continuous* if the boundary of one element is also the boundary of the next, whilst the row is *disjunctive* if two consecutive elements have separate boundaries.

§ 30. A surface, like a curve, may be *closed*, if, for instance, the generating curve is closed and returns into its original position ; or it may *extend indefinitely*, if we suppose the generating curve to move to an indefinite distance from its first position.

§ 31. If we now consider a surface of finite extension, and we move it about in space, it will either slide along itself or generate something new.

If a smooth piece of paper be placed on the flat surface of a table, it may be made to slide about that surface, and will thus have a surface as its path. Similarly the surface of a globe or of a cylinder may slide along itself.

But this is a very special motion, possible only with some special surfaces. In general the moving surface describes a solid.

§ 32. **Path of a Moving Surface.**—*The path of a moving surface is in general a solid.*

§ 33. If we fix on one position of the moving surface, the surface may move, in one *sense* or in the opposite sense, to other positions, as in the former cases. *A surface in a solid, therefore, has two sides.*

§ 34. **Dimensions of Space.**—Further, a solid is a one-way spread of the moving surface ; but, as the surface is a two-way spread of points, the solid is a

three-way spread of points. In other words, *a solid is of three dimensions.*

If we next move a solid about in space, it will be found that its path will be a solid and that we obtain nothing new. Space itself is a three-way spread of points, or *space is of three dimensions.*

That a solid has three dimensions is commonly indicated by saying that a solid has *length, breadth,* and *thickness*; but this applies strictly to some solids only. A globe can scarcely be called long or thick, nor can we very suitably speak of the breadth of a wire. The expressions are fully justified only for solids of the shape of a brick and similar shapes.

§ 35. The above considerations have made us acquainted with properties of space. These have been obtained by observation. Collecting them into one proposition, we obtain our first axiom—that is to say, a statement obtained by experience.

Axiom I. *Space is of three dimensions.*

Or, in the language of § 27 :

Space is a three-way spread, with points as elements.

This includes the statement that space is *continuous* (§ 29).

§ 36. **Figures.**—Solids, surfaces, lines or curves, and points, or any combination of them, will be called *geometrical figures,* or *figures* simply.

A figure has therefore in general all the properties enumerated in § 7. It has shape, size, and position, and it may be moved—that is, it may change its position without changing its shape or size. Shape and size, too, are in some investigations considered as liable to change, but in elementary geometry they are

generally treated as invariable. This, however, does not prevent us from speaking occasionally of moving points or lines in a figure, although the shape of a figure is thereby changed.

§ 37. **Coincidence of Two Figures.**—Two figures which have the same shape, the same size, and the same position are said to *coincide* or to be *coincident.* Two coincident figures cannot be distinguished from one another till they are separated by moving one into a different position. If both figures are now moved, each on a separate path, to any other part of space, we may try to make them again coincident. That we shall succeed in this could not be proved beforehand. But experience teaches that if we make one solid fit here and now into the hollow of another, it will also fit at any other time and at any other part of space. Thus the different parts of a machine which fit here, will fit if they are sent to India, and this equally whether they are sent all by the Suez Canal, or whether some are sent by this route and others round the Cape. If they do not fit, we say that one or other of the parts has changed its shape or size; and we account for it by changes in the *material* produced by bending, by change of temperature, or by some other physical cause. The only criterion which we have as to the invariability of shape and size consists in the fitting in of material solids into hollows made in other solids.

§ 38. This important result ot our experience we shall consider as applicable to geometrical figures, and thus obtain our second axiom.

Axiom II. *Figures may be moved in space without change of shape or size.*

The full meaning of this is :

Two figures which can be made to coincide at one part of space can be made to coincide at any other part, whatever be the path on which each figure is moved from the former position to the latter.

This axiom implies *that figures which are possible at one part of space are possible at every other part*, or *that space is everywhere alike.*

§ 39. **Definition of Congruence.**—*Two figures which can be made to coincide are said to be congruent.*

In many text-books on geometry such figures are called *equal in all respects*, or *identically equal.*

Congruent figures are equal in all respects excepting in position. To decide whether two figures are congruent we must try to make them coincident, or we must, as this operation is called, *apply* the one to the other.

Figures which agree in shape, but not in size, are called *similar*, and figures which agree in size, but not in shape, are called *equivalent.*

Figures which are congruent are, therefore, *similar* and *equivalent.*

In the present treatise congruent figures only will be treated of.

§ 40. Experience shows not only that solids may be moved, but also that such motion is possible if one or two points of the solid be fixed.

If a weight be suspended by a string it can swing about freely. As long as the string remains stretched we may consider the weight together with the string as forming a single solid of which one point, the fixed end of the string, is immovable.

If the weight is suspended by two strings, like the seat of a

swing, as long as both strings remain stretched we may again consider the whole as one solid, of which two points are fixed. It will still be able to move, and, when turned quite round, will come back to its first position.

Assuming this property to hold for all geometrical figures, we obtain another axiom, namely,

Axiom III. *A figure with two points fixed can still be moved, but only in one way, though in either sense, and will, if moved far enough in either sense, return to its original position.*

CHAPTER II.

LINES AND PLANES.

§ 41. **The Line.**—If we suspend a weight by a string, the string becomes stretched, and we say it is straight, by which we mean to express that it has assumed a peculiar definite shape. If we mentally abstract from this string all thickness, we obtain the notion of the simplest of all lines, which we call a straight line.

We may suppose the string to go over a pulley, and then, on allowing more string to pass over, the weight would move downwards. Thus the stretched part of the string would become longer and longer, till the weight reached the ground. Now we may suppose that the pulley is raised, or that the weight sinks into a hole in the ground; thus we might *lengthen*, or *produce* the straight part of our string

both ways. And this process, though practically we should soon find it impossible to go further, may be conceived to be continued to an indefinite extent. If now the string became rigid, so that we might take it out of its original position without changing its shape —that is to say, without destroying its straightness— and if at the same time the string lost all thickness, it would give us the notion of a straight line.

§ 42. A straight line will in future be called a *line* simply. All other lines will be called *curved lines*, or *curves.*

We have obtained the following property :—

A line is of *indefinite extension*—that is, a point which describes it may move along it, in either sense, to a greater and greater distance, until it is lost in unknown regions, of which our experience does not teach us anything.

The word 'line,' when used without qualification, will always denote the line in its entirety, that is, extending indefinitely in both directions.

§ 43. According to Axiom II., we may move our line freely in space. We may move it so that one point in it is brought to any given position, and we can thus make a line pass through any point A given arbitrarily in space.

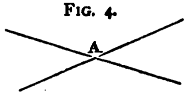
If this point A remains fixed, we may still, according to Axiom III., turn the line about it. The line will then sweep through space, and, while always passing through A, will pass through fresh points besides A at every change of position ; and we can easily convince ourselves that it can be made to pass through any second point B, given also arbitrarily

anywhere in space. At least between any two points within our reach we may stretch a string, hence we may draw a line between them.

We thus come to the conclusion that a line may be placed in space so that it passes through any two given points, or *that through any two given points always at least one line may be placed.*

§ 44. We may of course obtain as many lines as we like. Suppose we take two. They may have any positions in space, and do not necessarily meet. But we may move them so that they both pass through a point *A*. In this case we say that the lines *meet*, or *intersect*, or *cut each other.*

FIG. 4.



We may further move both lines so that they pass through two points, *A* and *B*. If this is done, our two lines have two points in common.

Our notion of straightness suggests at once that these two lines have not only the two points *A* and *B* in common, but all others besides ; that the lines, in fact, coincide throughout their length. Reference to the strings, as rough representations of the lines, cuts off every other assumption. For if we stretch two strings, and keep them close together at two points, they will fall together throughout their length. We suppose that our two lines will have the same property, that is, that two lines which have two points in common have all points in common.

§ 45. We have thus obtained properties of lines by abstraction from observation. If we state them in form of a proposition this must be taken as an axiom.

Thus we obtain the following axiom :—

Axiom IV. Of the Line. *Through t always one, and only one, line can be drawn.*

This statement includes all our previou relating to the line. It is, in fact, only anot of saying that if we draw two lines through two these lines will fall together, so as to form single line ; or

Two lines which have two points in common cide throughout their indefinite extension, that is t every point in the one, both *between* and *beyond* two first points, coincides with a point of the othe

Making use of the terminology of § 39, we may that lines are *congruent*, and that they are coincid if two points of the one coincide with two points the other.

This includes the following theorem :

Theorem : *Two different lines cannot have mor than one point in common.*

Two lines have not, however, necessarily a poin in common.

§ 46. As all lines are congruent, we may conside them all as copies of one another. Thus, if we want t see whether any curve is straight, that is, whether it i a line or not, we try to make it coincide with a line In order to draw lines on paper we use the edg of a ruler or straight-edge, and move the point of a pencil along it. The pencil line thus obtained is of course, material, and has breadth and thickness however fine we draw it. It gives us, therefore, not a geometrical line, but only a representative, like th string which we used before as an illustration.

Geometrical lines have no thickness. Hence if w

place any number of lines through two points they all coincide and give only a single line, which still has no thickness, however many lines are put together—just as we saw before that any number of points may be brought together without giving anything more than a point.

If we wish to test the straightness of our straight-edge, we draw by its aid a line on paper—that is, we take a copy of it—and then see whether the edge of the straight-edge coincides everywhere with the line whenever it is made to coincide with it at two points. The straight-edge should be tried in this way against each side of the line drawn. If the edge coincides with the line now, it is straight.

§ 47. **Join of Points and of Lines.**—The line joining two points is called the *join* of the two points. The point common to two intersecting lines is called the *join* of the two lines.

§ 48. **Pencil of Lines.**—A fixed point A may be joined to all other points in space. We get thus all the lines which can be drawn through the point A. The aggregate of all these lines is called a *pencil of lines*, or a *pencil of rays*, the lines being in this case often called rays. The fixed point is called the *centre*, or the *base*, of the pencil. Any one of these rays is said to be a line in the pencil, and also to be a *line in the fixed point*. In this sense we say, not only that a point may lie in a line, but also that a *line may lie in a point*, meaning that the line passes through the point. This mode of expression, or the use of the same phrases in reference to lines and points, will be seen hereafter to be particularly convenient.

§ 49. **Rotation about an Axis.**—According to Axiom III., any figure may be moved if two points of it, *A* and *B*, be fixed. If we now take a line joining *AB*, and turn a figure about these two points, the different positions of the line *AB* will all be lines passing through *A* and *B*, and will therefore all coincide.

Hence, if a figure turns about two fixed points, the line joining them will remain in its original position, and the motion of the figure will be the same if, instead of the points originally fixed, we fix any other pair of points, or even all the points, in the line joining them.

This motion is called *rotation*, and the fixed line the *axis of rotation.*

§ 50. **The Plane.**—Just as the straight line is simpler than any curved line, so a flat surface is simpler than any curved surface.

The notion of a flat surface we again obtain by observation ; the surfaces of walls and of many pieces of furniture are, roughly speaking, flat. The still surface of a lake, or the surface of a well-polished looking-glass or flat mirror, gives even a better idea of flatness.

A flat surface is in geometry generally called a *plane surface*, or a *plane.*

§ 51. The looking-glass, as well as the lake, is limited in extent, and gives us therefore the notion of a plane which is also limited. But we know that looking-glasses are of different size, and if we have a particular one before us we may imagine a bigger one without difficulty. We thus imagine a plane greater than the one directly suggested by the surface of the *glass ; and to this* our imagination there is no limit.

This leads us to consider a plane as being of *unlimited extension* in all directions.

A plane thus divides space into two parts, one part on each side of it, the one below and the other above the plane, or the one in front and the other behind, or the one to the right and the other to the left, &c. These two parts of space are completely separated by the plane, so that a point which moves from one side of the plane to another must necessarily pass through the plane. The plane is *continuous.*

§ 52. If we now conceive a plane, that is, a flat surface of unlimited extension, we may, according to Axiom II., move it freely through space, and may do this until a point on it comes to a point A which has been chosen anywhere in space. It is then said that the plane passes through A. If we keep A fixed we may turn the plane about it, and may do so until the plane comes to pass also through a second fixed point B, likewise chosen arbitrarily in space. According to Axiom III. and § 49, we may still move the plane, as only two points of it are fixed, by turning it about the line joining them, and this turning may be continued until the plane passes through a third point C, chosen arbitrarily, like A and B. Then our plane will, as a rule, be fixed. Thus it appears that we may place a plane so as to pass through three points, A, B, C, chosen anywhere in space.

But if C happens to lie on the line joining A and B, then a plane through A and B, which did not pass through C, could never be made to pass through C by being rotated about A and B; for if it did contain C *in one position,* it would contain it in all positions, as

this point would remain fixed during rotation. ought, therefore, to limit the conclusion arrived follows :

Through three points *which do not lie in a lin* may always pass a plane. Whether a plane ma drawn through three points which do lie in a li remains for the moment an open question.

§ 53. If we take two flat mirrors and put the polished faces one on the other, if they are well mad their surfaces will touch throughout—that is to say, every point in the one will coincide with one point in the other as far as both extend together. The one may further be made to slide along the other without destroying this coincidence.

This suggests a new property of planes—namely, that on placing two planes one on the other, they may be made to coincide ; or, *that all planes are congruent.*

According to § 52 any plane may be placed through any three points which do not lie in a line, and then its position is fixed. Since planes are congruent it follows that, if we place two planes through the same three points, they may coincide throughout ; and it will become evident that they must do so, if we consider the result of attempting to make three points in the face of one mirror fall on the face of another mirror. Experience thus leads us to the following conclusion, which we state in the form of an axiom, thus :

Axiom V. Of the Plane. *Through three points which do not lie in a line, one, and only one, plane may always be drawn.*

This may also be stated thus :—*Two planes coincide throughout their indefinite extension if three points in*

the one, not in the same line, coincide with three points in the other; or thus:

Three points not in a line are necessary and sufficient to determine a plane.

§ 54. **Intersection of two Planes.**—If two planes have two points, *A* and *B*, in common, they must necessarily have more points in common. For, since each extends continuously without limit, a point moving in the one plane through the point *A* or *B* will cross the other plane at this point (comp. § 51); hence one plane will lie partly on the one and partly on the other side of the second plane. They must therefore intersect.

The intersection of two planes must be a line, for if three points common to both the planes do not lie in a line, the two planes will be coincident, and this we do not suppose. Hence

Theorem: *The intersection of two planes is a line.* This line is called the *join* of the two planes.

§ 55. As the points *A* and *B* are common to both planes, it follows that the line of intersection must be the line joining *AB*. Hence the line *AB* lies in each of the two planes. But the two points *A* and *B* may be taken anywhere in the first plane, and thus we see that

Theorem: *A line which has two points in common with a plane lies altogether in that plane.*

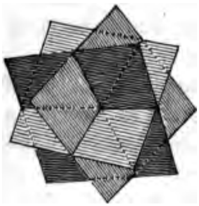
§ 56. **Axial Pencil.**—Through a line we can thus draw an unlimited number of planes, namely, all the planes which pass through two points in that line.

The aggregate of all these planes forms what is called an *axial pencil of planes*, or an axial pencil

simply, the common line being the ***axis*** of this pencil.

§ 57. **Join of Line and Plane.**—*A line which does not lie altogether in a plane cannot have more than one point in common with the plane*, for if it had two in common it would lie in the plane. This does not, however, imply that every line has a point in common with every plane, but if a line meets a plane it meets it in one point only. This point is called the ***intersection*** or the ***join*** of the line and plane.

Fig. 5.



§ 58. From this, again, it follows that *three planes which do not pass through the same line cannot have more than one point in common*; for the points common to two planes lie on a line, and this line can have only one point in common with the third plane.

§ 59. *A line and a point without it determine a plane.* For, any two points in the line together with the given point determine one plane, which according to § 55 and by its construction contains the line and point.

§ 60. The most important properties, as far as the following investigations are concerned, are, *first*, that all planes are congruent, hence that properties proved for one plane hold for all; and *secondly*, that lines may be drawn in a plane: or more precisely, that a line which joins any two points in the plane lies altogether in the plane.

It is therefore possible to draw in a plane figures which consist of any number of lines. Of course curves may also be drawn in a plane.

Figures drawn in a plane are called *plane figures.*

§ 61. The study of figures in a plane constitutes *plane geometry*, and it is with this branch of geometry that we shall be concerned in this work.

EXERCISES : *Drawing.*

Materials.—Paper, straight-edge, and pencil. The paper used ought to be moderately rough, to take the pencil-marks easily.

The pencil must be rather hard and well pointed, or better sharpened on a fine file to a chisel-like edge. Such an edge does not require as frequent resharpening as a point. For writing in reference letters a softer pencil, pointed, may be used.

(1) **Test of Straight-edge.**—In order to see whether the edge is straight, draw a line along it on the paper, and place the edge from the other side against the line. If the edge again coincides with the line, the edge is straight.

(2) Take two points A, B, and draw their *join c.*

The join of two points does not terminate at the two points; hence in your drawing the join of A, B should be produced both ways beyond A and B. (A point is marked best by two short strokes crossing each other, or by a prick with a needle, but not by a dot.)

(2*a*) Draw two lines a, b, and mark their join C.

Such a join may, of course, fall outside your piece of paper, unless the lines be properly chosen. In more complicated figures, such as are contained in the following exercises, it will be found that it is not always easy to get all those joins of the lines on the drawing-paper which are essential in the figure. In these cases another trial should be made.

(The join of two lines is marked by drawing a small circle round the point, not by a big dot.)

(3) Take three points A, B, C, and draw their joins two and two. The figure is called a *Three-point.*

(3*a*) Take three lines a, b, c, and mark their joins A, B, C. *The figure is called a Three-side.*

(4) Take four points A, B, C, D, and draw the join of any pair of them. Show that there are six of these joins. The figure is called a *Four-point.*

(4*a*) Take four lines a, b, c, d, and mark the join of any pair of these. Count these points; there are six. The figure is called a *Four-side.*

(5) In Ex. (4) mark the joins of any two of the six lines obtained which are not the given points.

(5*a*) Do the same with the lines joining the points in Ex. (4*a*).

(6) Draw any figure you like consisting of points and lines, and together with it a second figure, according to the following law. Whenever you take in the one figure a line, take in the other figure a point, and for every point in the first take a line in the second. Where you take in the one 'a line and a point in it,' or 'two points and their join,' take in the other 'a point and a line through it,' or 'two lines and their join.'

Figures related to each other in this manner are said to be *reciprocal.* Simple examples have been given in previous exercises.

(7) Draw three lines p, q, r through a point S. On each take two points: viz. A, A' on p; B, B' on q; C, C' on r. Join BC by a line a, CA by a line b, AB by a line c. Similarly join $B'C'$ by a', $C'A'$ by b', and $A'B'$ by a line c'. Mark the join of a and a' as P, of b and b' as Q, of c and c' as R. If your drawing is accurate these three points P, Q, R must lie on a line s, that is, the line PQ must pass through R.

(8) Draw the figure reciprocal [see exercise (6)] to the last.

(9) Take a line s and three points A, B, P on it. Through A draw any two lines a and a'.

Through P draw any one line p. Mark the points (ap) and $(a'p)$. Join B to (ap) by a line b, and to $(a'p)$ by a line b'. Mark the points (ab') and $(a'b)$, and draw their join q. Mark the point where q cuts the given line s, and call it Q.

Repeat the construction, only taking for any or all of the lines a, a', p other lines through A and P. If your drawing is correct the last line will again pass through Q.

(10) Draw the figure reciprocal to the last.

DIGRESSION ON LOGIC.

[1] A *Proposition* is a sentence which puts before us or states a fact.

Thus 'Space is a fundamental conception' is a proposition. It consists of three parts. The first is 'space,' the third is 'a fundamental conception.' These are connected by the word 'is' (called the *copula*), and thus the proposition is formed.

The meaning of a proposition will be better understood after the following remarks.

[2] We may divide all things or notions into *classes*, according to their properties.

A class may be described by a statement of the properties which enable us to decide whether a thing belongs to it or not.

Such a statement is called a *definition.*

Very often a class has a special *name.* In fact, nouns and adjectives are names of classes. Thus *man* is the name of a class, so is *white.* Some classes are denoted by compound expressions, as 'white man,' or 'boundary of a solid;' others by stating their definition. We denote for the present a class by a letter, such as A, B, C, or by class A, class B, class C.

[3] A class may consist of only one single individual. 'Queen of England' is such a class; 'geometry' and 'space' are others.

[4] If we denote by A the class of things possessing any given property, all other things not possessing this property form another class, which is called the *negative* to the first, and which is denoted by '*not A*.'

Any one thing belongs either to the class A or to the class 'not A,' *but no thing belongs to both.* It follows that not only is '*not A*' the negative of A, but that also A is the negative of '*not A*.'

The class A and the class 'not A' *are mutually the negatives of each other, and both together include all things in the universe.*

In most speculations, however, we do not treat of all things in the universe, but only of those belonging to a certain class.

We thus treat that class as our universe, leaving out of consideration everything that does not belong to it.

Within this chosen class we form again the classes '*A*' and '*not A*,' and call them still mutually the negatives of each other, though they do not now apply to the whole universe. Thus in a treatise on colour it matters little whether or not we include music, or motion, or mathematics in the class '*not red*.' And in geometry, where we have to deal with geometrical things only, it would be needless to state explicitly that we do not include in the class '*not triangle*' stars, or trees, or telephones, or anything which is not a geometrical figure. Though this is only partial negation, and ought to be distinguished as such, as by speaking in the above examples of 'colour not red' and of 'figures which are not triangles,' instead of saying merely 'not red,' 'not triangles,' all rules given here will hold for it, and it will not be necessary for our purpose to state them specially for such cases. But it has to be remembered that in the following pages '*not A*' may mean partial negation in the sense just explained.

[5] A proposition may be expressed in the form

$$A \text{ is } B \qquad (1).$$

This says that, if a thing belongs to class A, then it also belongs to class B.

If this is true, then every individual A belongs to class B; hence no A belongs to the class '*not B*,' or no thing '*not B*' is a thing A; that is, every '*not B*' is a '*not A*.'

The propositions

$$A \text{ is } B \qquad (1),$$

and

$$\text{'}not\ B\text{' is '}not\ A\text{'} \quad (2),$$

express thus the same fact, and are only different forms of the same proposition.

Definition.—*The propositions* (1) *and* (2) *are called each the contra-positive form of the other.*

Though both forms express the same fact it is nevertheless often of importance to consider both. The contra-positive form often puts the truth expressed in a different light, so that the full

meaning of the statement made may be more easily comprehended.

[6] The statement *A is B* means that every individual belonging to the class *A* also belongs to the class *B*. The class *B* thus contains all the individuals of class *A*; that is to say, some of the *B* are *A*. It may, however, happen not only that some but that all individuals of *B* belong to class *A*.

In other words, if we know that *A is B* we also know that

some B is *A*,

whilst it may possibly happen that

all B is *A*,

or, more shortly, that

B is *A* (3).

This proposition *may* be true, but it is not a *logical* consequence of the given proposition that *A is B*.

Definition.—*The Propositions*

A is *B* (1)

and *B* is *A* (3)

are called each the converse to the other.

The contra-positive of the converse (3), *namely,*

'not *A*' is 'not *B*' (4),

is called the obverse of the original proposition.

[7] From the truth of either of two converse propositions, that of the other cannot be inferred. If, however, both are true, the classes *A* and *B* are *identical.*

If proposition (1) is true, the proof that a thing belongs to class *A* also proves that it belongs to class *B*. In other words, if *A is B*, then in order to know that a thing is *B* it is **sufficient** to know that it is *A*, but it is not **necessary** that a thing must be *A* in order to be *B* unless *A* and *B* are identical.

We thus see that the three Propositions:—

1. 'The class *A* and the class *B* are identical;'

2. 'The proposition *A is B* and its converse *B is A* are *both true;*'

3. 'In order to prove that a thing is *B* it is *necessary* and *sufficient* to show that it is *A*'—
are only different forms of the same proposition.

[8] *Example* I. Let us take as the original proposition the following from § 16:

'The path of a moving point is a curve.' The contra-positive is, 'A thing which is not a curve is not the path of a moving point.'

The converse is—

'A curve is the path of a moving point,' of which the contra-positive form is, 'Anything which is not the path of a moving point is not a curve.'

In this case it follows from §§ 16, 17 that the converse proposition is true, for a point may be moved along every curve. The class 'path of a moving point' and the class 'curve' are identical.

If we treat in the same way the proposition in § 23,

'A surface is the path of a moving curve,'

the converse is—

'The path of a moving curve is a surface.'

Here the converse proposition is not always true, for, as has been pointed out in § 22, the path of a moving curve is not always a surface. Hence the class 'surface' and the class 'path of a moving curve' are not identical.

The same is true of the proposition in § 32:

'The path of a moving surface is, in general, a solid.'

[9] *Example* II. Let us consider the proposition Axiom I.:

'Space is of three dimensions.'

In this form the two classes are 'space' and 'thing of three dimensions.'

The second class may be better defined if we use the second orm of the Axiom:

'Space is a three-way spread with points as elements.'

The contra-positive of this is—

'Anything which is not a three-way spread with points as elements is not space.'

This includes, as a special case—

'A three-way spread which has not points as elements is not *space.*'

The converse is—

'A three-way spread with points as elements is space,'

Or in its contra-positive form—

'Anything which is not space is not a three-way spread with points as elements.'

This proposition is not true, for three-way spreads with points as elements are conceivable with properties very different from those of space. Hence the classes 'space' and 'three-way spreads with points as elements' are not identical.

[10] We have to deal in Geometry with different kinds of propositions, viz. with Definitions, Axioms, Theorems, Corollaries, and Problems. What a Definition is we have seen already, viz.

A proposition which states the properties that enable us to decide whether a thing belongs to a class or not is called a **Definition** *of the class.*

A good definition must state as many properties as are *sufficient* to decide whether a thing belongs to a class or not, but not more than are *necessary* for this purpose.

[11] We have in this book only to treat of classes relating to Geometry, principally of figures. Whenever we introduce a new figure we have to define it. This can only be done by reference to simpler figures out of which it is formed. These simpler figures, again, are defined by aid of still simpler ones, till at last we arrive at the simplest figures possible, which from their very simplicity cannot be defined geometrically. These are *point, curve, surface, solid.* We have notions of these, but we cannot give true definitions of them, at least not at the beginning of our study. The same holds for *line, plane,* and *space.* We may say (§ 50) 'a flat surface is called a plane,' and call this a definition of a plane. But this defines the class 'plane' by aid of the class 'flat.' What 'flat' is remains undecided. But then there is the Axiom V., which tells us what we do understand by *flat.* It enables us, if we apply it to a surface and a copy of it, to decide whether a surface is a plane; and thus it takes the place of a definition. The same is true for the other Axioms. *All six Axioms together define space.*

The first tells us that space is a *spread.* To distinguish it from other spreads we learn that it is a *three-way* spread, and

that it has *points* as elements. The second says that the three ways in which we may go from one element to others are of the same kind; that they are not different, as in the example given in § 28 about the two ways of changing a musical tone. The true nature of the remaining axioms cannot be explained at this stage. It may be here sufficient to state that many three-way spreads of points are conceivable, and that the axioms state which of these conceivable ones has the properties that are observed in space.

[12] *A* **Theorem** *is a proposition the truth of which is inferred from other propositions that are known to be true. The process by which this truth is established is called the* **Proof** *or the* **Demonstration** *of the Theorem.*

A Corollary to a Theorem is a proposition whose truth follows immediately from that of the Theorem.

A **Geometrical Theorem** always relates to some *Figure*, and says that a figure which has a certain property has of necessity also another property. To bring it to the form of a simple proposition A *is* B, we may state it thus:

'A figure which has a certain property' is 'a figure which has another specified property.'

The first part contains the name or definition of the figure to which the Theorem relates, and is called the *Antecedent* of the Theorem. The second part contains another property not stated in the antecedent, and is called the *Consequent.*

[13] In order to understand the meaning of a Theorem, the student should always at first make out clearly the nature of the figure about which a statement is made, and should, if no figure is provided, draw one for himself.

This is advisable even if a figure is inserted in the book, for nothing affords so much assistance in understanding the importance of the different properties which a figure possesses, and in realising how its different elements (lines and points) are related to one another, as seeing how they come into existence one after the other. The student should begin by finding out which parts of the figure are given arbitrarily, that is to say, which lines he may draw as he likes or which points he may assume where he likes. Then follow other parts which have a definite and prescribed relation to these.

[14] From what has been said about Propositions it follows that of two Theorems which are one the contra-positive of the other, only one needs a *geometrical proof*; the other is a *logical consequence* of it.

The *converse* and the *obverse Theorems*, on the other hand, are not logical consequences of the original, and one of them, therefore, *always* requires a special proof. This takes different forms, of which the more important ones may be enumerated.

1. If it has been proved that every A is B, and also that there is but one individual in the class B, then we infer that B *is* A.

The geometrical proof required for the converse is here contained in the proof that there is but one B. This shows the importance of proving that there is but one in a class.

2. If it has been proved that every A is a B, that no two A's are the same B, and that there are as many individuals in class A as in class B.

For example, if you see ten soldiers in a room with a number of swords hanging on the wall, and know that every soldier owns one of the swords, then conversely you also know that every sword belongs to one of the soldiers, if you find on counting that there are but ten swords.

3. In many cases the converse theorem or its contra-positive, the *obverse* of the original, has to be proved quite independently of the original theorem.

[15] In a **Geometrical Problem** it is generally required not to find or prove properties of a given figure, but to find a figure which shall have certain prescribed properties.

The determination or construction of such a figure is called the **Solution** of the problem.

When the solution has been obtained, a Proof is required that the solution is correct. This proof in many cases is contained in the reasoning which has led to the solution. In other cases the solution has been obtained by intuition or by a happy guess, and in these cases the *proof* must not be omitted.

Exercise.—*Determine the logical connexion between Axiom V., § 53, and the theorems in* §§ 54, 55, and 57.

CHAPTER III.

SEGMENTS AND ANGLES.

§ 62. **Reciprocal Figures.**—Every figure in a plane consists of a combination of points and lines, and of curves generated by the motion of a point or by the motion of a line. Points and lines are, therefore, called the *elements* of plane figures.

At first sight it might appear that points only deserve this designation, but we shall soon see that lines are of equal importance in the generation of figures. In many cases we consider points only as elements, in others lines only, in others again both points and lines. In most cases we can, when one figure is given, construct another such that lines take the place of points in the first, and points the place of lines. Any theorem concerning the first thus gives rise to a corresponding theorem concerning the second figure. Figures and theorems related in this manner are called *reciprocal figures* or *reciprocal theorems.*

§. 63. Let us suppose, as a simple example, a figure consisting of two points, A, B (fig. 6); the reciprocal figure will then consist of two lines, a, b (fig. 7). The first figure has the property that there is one line, c, joining the two points. Corresponding to this we find in the second figure the property that two lines a, b have a point C in common. Hence we get as propositions, corresponding in the manner indicated, the following:

FIG. 6.

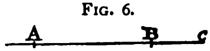
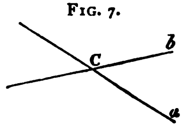


FIG. 7.



Two points have a line in common; or, *the join of two points is a line.* And reciprocally—

Two lines have a point in common; or, *the join of two lines is a point.*

§ 64. The first of these propositions is always true; the second, though it is generally true, is not always so, for we shall soon see that two lines in a plane do not necessarily intersect. This, however, is an exception, but it has to be remembered.

The fact that a proposition is true as a rule, but with some exceptions, is expressed by the words '*in general.*' Hence whenever these occur in any proposition it is to be understood that there are *exceptions* to the proposition, but so that among an *infinite* number of cases in which the proposition is applicable there are only a *finite* number of exceptions. The last proposition ought, therefore, to be stated more exactly:

Two lines have in general a point in common, and, *if two lines have a join, this is a point.*

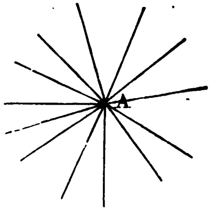
§ 65. In all that follows points will be denoted by capital letters, A, B, $C \ldots P$, $Q \ldots$ and lines by small letters a, b, $c \ldots p$, $q \ldots$ The join of two elements will be denoted by putting the letters, indicating the elements, together. Thus, the line joining the points A and B is called the line AB, whilst ab denotes the point of intersection of the lines a and b.

§ 66. **Row of Points, Pencil of Lines.**—A line contains an infinite number of points, which, according to § 19, form *a row of points on the line*; but, for shortness sake, this will be called in future simply *a row of points*.

The line containing the points is called the base *of the row.*

As the figure reciprocal to a row, viz. all points in a line, we get *all lines in a point*, or, in common language, all lines passing through a point.

FIG. 8.



The aggregate of all lines in a plane which pass through a given point is called a *pencil of lines in a plane*, or a *flat* pencil, to distinguish it from the pencil of lines in space (§ 48). In plane geometry, where only figures in a plane are considered, it is sufficient to call it simply a *pencil of lines*, or a *pencil of rays*.

The common point is called the *centre* or the *base* of the pencil.

We have thus the reciprocal propositions

A point *moving* along a line describes a row.

A line *turning* about a point describes a pencil.

In either case the motion, or turning, may be effected in one or the opposite *sense*. (§ 17.)

Sometimes it is convenient to consider each line in the pencil as terminated at the centre, and as described by a point moving from the centre in one way to an indefinite distance. The line is then called a *half-ray*.

§ 67. **Direction.**—Two lines or half-rays drawn through the same point are said to have the same *direction* if they coincide; otherwise they are said to have *different* directions. Two half-rays, of which one is the continuation of the other, are said to have *opposite directions*.

Two half-rays which have opposite directions lie,

therefore, in the same line, and are of opposite *sense* in this line.

§ 68. **Segment.**—If two points, A, B, be taken in a line, then that part of the line which is bounded by A and B is called (§ 21) a *segment of the line*, or where no ambiguity is possible, simply a *segment*. The two points A and B are called the *end points* of the segment. The segment is denoted by AB.

§ 69. Two segments, AB and CD, can always be placed one on the other in such a manner that C falls on A, that the two lines of which they are segments coincide, and that B and D lie on the same side of A. In this position the point D falls either between A and B, or on B, or beyond B. A point moving from A along the line will either first reach D, and afterwards B, or it will reach B and D at the same time if these coincide, or it will reach first B and then D.

In the first case it is said that CD is less than AB, in the second that CD is equal to, or of the same length as, AB, and in the third that CD is greater than AB. This is expressed in symbols as follows:

$CD < AB$	means	CD is less than AB,
$CD = AB$	,,	CD is equal to AB,
$CD > AB$	,,	CD is greater than AB.

These expressions relate to the *length* of the segments.

§ 70. **Measurement of Segments.**—When any two segments are given, *one*, and *only one*, of these three relations must necessarily hold good. To decide which exists, the above criterion requires that we *place the one* segment on the other. This operation

is practically seldom possible. If, for instance, both segments are drawn on the same paper, we cannot actually move the one line towards the other unless we cut the paper. We must, therefore, use a different method. This consists in taking a third segment, PQ, movable in space, which we compare first with the one, AB, and afterwards with the other, CD. If we take this third segment PQ equal to AB, and find on moving it to CD that it is also equal to CD, we conclude that CD equals AB. For if we suppose that the segment PQ is made to coincide with CD, and if we move both together to AB, keeping them coincident, then on making PQ coincident with AB, to which it was supposed equal, CD will also coincide with AB. (Axiom II. § 38.)

This is generally expressed by saying, two magnitudes *which are each equal to a third are equal to one another*; and the statement is taken as an axiom. We see that it is in the present case a consequence of our Axiom II., which relates to the movability of figures without change of shape or size. It may also be taken to be a definition of equality of length of segments, or rather to be a criterion.

All our measurements of length depend upon this proposition.

§ 71. For geometrical purposes this is done generally by aid of a pair of compasses. Their two points, which we may call P and Q, can within certain limits be opened to any distance required, so that they can be made to coincide with the end points of the segment AB, and then a segment PQ, equal in length to AB, may be carried about and compared with the nent CD.

A pair of compasses may thus be said to be an instrument for carrying distances, or segments of given length, about in space.

§ 72. A segment having the end points A and B may be supposed described by a point moving either from A to B or in the opposite sense from B to A. These two segments, which are equal in length but of opposite *sense*, are distinguished as AB and BA respectively, so *that* AB *means the segment described by a point moving from A to B.*

§ 73. If two different segments, AB and CD, lie in the same line p, they may be compared as to sense. They have the same or opposite sense, according as points moving from A to B and from C to D respectively move in the same or in opposite sense in the line p. Or we may compare them thus:

We suppose the segment CD to slide along the line p, till C coincides with A; then D will fall either at D', on the same side of A as B (fig. 9), or at D'' on the opposite side of A (fig. 10).

In the first case the two segments are of the same sense, in the other case they are of opposite sense.

Fig. 9.

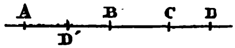


Fig. 10.

§ 74. Two segments in the same line which are equal in length and of the same sense will in future be called *equal* simply, while, if they are equal in length and of opposite sense, they will be called *equal and opposite.*

It is convenient to indicate the sense also by a *sign.* For this purpose the signs + (read plus) and

— (read *minus*) are used, which are borrowed from algebra. If AB and CD are equal in length and of the same sense, or simply equal, this is indicated by writing

$$+AB=+CD;$$

but if AB and CD are equal in length and of opposite sense, we write

$$+AB=-CD.$$

Generally the sign + is omitted, so that the equation

$$AB=CD$$

indicates not only that the two segments are equal in length, but also of the same sense, whilst

$$AB=-CD$$

means that AB and CD are segments equal in length but opposite in sense. This extension of the meaning of the sign = must be kept well in mind. At present this distinction of sense has a meaning only if both segments lie in the same line.

§ 75. The symbol AB thus means the segment described by a point moving from A to B, or AB is the result obtained by moving a point from A to B. $-AB$, on the other hand, means the same segment described in the opposite sense, that is, described by a point moving from B to A, so that

$$BA=-AB.$$

This equation expresses the fact that the two segments AB and BA are equal in length but opposite in sense.

Theorem : *If the order of the two letters indicating the end points of a segment be changed, the segment changes its sense.*

§ 76. **Sum of Segments.**—If A, B, C be three points in a line, then the segment AB is obtained by moving a point from A to B, and the segment BC by moving a point from B to C. If both these operations be performed in succession, we obtain a segment described by moving a point first from A to B and then from B to C, which is equivalent to moving a point from A to C, describing the segment AC. This is expressed by saying AC is the *sum* of AB and BC; in symbols

$$AB+BC=AC.$$

This definition of the sum of two segments does not suppose any definite order for the points A, B, C. It holds equally in the case where B lies between A and C (fig. 11), in which case the moving point describes both segments AB and BC in the same sense; or where B lies beyond C (fig. 12); or where B and C lie on opposite sides of A (fig. 13). In the latter cases the moving point describes the second segment in a sense opposite to that of the first. But the above equation holds always.

FIG. 11.

FIG. 12.

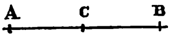


FIG. 13.

§ 77. If $A, B, C, D \dots G, H$ are points in a line, we get in the same manner

$$AB+BC+CD+\dots+GH=AH.$$

This means, if a point moves from A to B to C to ...to G to H, it is at the distance of AH from the starting-point A, whatever the positions of the points.

§ 78. If we take the case where the last point

coincides with the first, the sum is equal to AA. But AA has no length. This is expressed by saying AA equals zero :

$$AA=0.$$

Applying this we get the important formulæ

$$AB+BA=0,$$
$$AB+BC+CA=0,$$
$$AB+BC+CD+\dots+GH+HA=0,$$

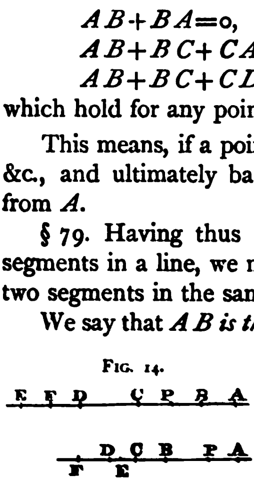
which hold for any points in a line.

This means, if a point moves from A to B to C to &c., and ultimately back to A, it is at no distance from A.

§ 79. Having thus defined the sum of *consecutive* segments in a line, we may now define the sum of any two segments in the same line.

We say that *AB is the sum of two segments CD and EF in the same line; if a point P can be found such that AP and PB are equal respectively to CD and EF,* taking, of course, account of the sense.

FIG. 14.



In symbols

$$AB=CD+EF,$$

if a point P can be found such that $AP=CD$ and $PB=EF$. Similarly for the sum of more than two segments.

§ 80. **Difference of Segments.**—*By the difference of two segments AB and CD, we mean a segment EF, which, when added to CD, gives AB as sum.*

Denoting the difference of AB and CD by $AB-CD$, we have

$$AB-CD=EF \text{ if } EF+CD=AB.$$

From this, since $AB+BC=AC$, it follows at once that

$$AC-BC=AB,$$

and also that $$AC-AB=BC.$$

In words:

The difference between two segments with common end-point equals the segment described by a point moving from the initial point of the first to that of the second;

The difference between two segments with common initial point equals the segment described by moving a point from the end point of the second to that of the first.

As a special case we get

$$AC-AC=0,$$

for it is equal to AA.

The difference of equal segments is zero.

§ 81. We have obtained the two equations

$$AC+CB=AB \text{ and } AC-BC=AB.$$

Hence $$AC-BC=AC+CB.$$

But CB and BC denote the same segment taken in an opposite sense. If we use the sign $-$, as before, to indicate negative sense, so that $BC=-CB$, we may write the above equation

$$AC-(-CB)=AC+CB.$$

Hence the proposition

Instead of subtracting a segment we may change its sense and add it.

§ 82. By definition $AB-CD$ is a segment which, when added to CD, gives AB, or

$$AB-CD+CD=AB.$$

If we write here $-DC$ for CD and $+DC$ for $-CD$, we get

$$AB+DC-DC=AB.$$

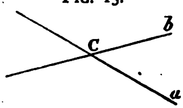
Here DC may be any segment whatsoever.

This shows that *if the same segment be first added to, and then subtracted from, or first subtracted from and then added to, any given segment, we obtain the original segment*; or addition and subtraction, as defined in the above, are operations which are opposites to one another: the one undoes what the other does.

§ 83. The results obtained in § 82 show the complete analogy between this addition and subtraction of segments, with the corresponding operations in algebra, and also the analogy between the sign of a number in algebra and the sense of a segment in geometry, thus justifying the use of the same symbols.

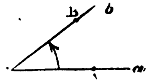
§ 84. **Angles.**—Having thus treated of segments, we have next to investigate the corresponding properties of the figure reciprocal to a segment. A segment was defined in § 68 as a part of a row, described by the motion of a point from one position A to another position B. The figure reciprocal to a row is a pencil (§ 66). Hence as the figure reciprocal to a segment we get a part of a pencil, described by the turning about the centre C of a line in the pencil from one position a to another position b. The result we call an *angle*.

FIG. 15.



It will, however, be convenient to consider at first a pencil of half-rays only. We then get the definition:

FIG. 16.

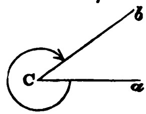


Definition of Angles.—*The part of a pencil of half-rays, described by a half-ray on turning about its end point C from one position* a *to another position* b, *is called an angle. The centre C of the pencil is called the* VERTEX, *and the first and last positions* a *and* b *of the describing ray are the* LIMITS *of the angle.*

The angle itself is denoted by $a\,b$, or, if A and B are points on the limits a and b respectively, by $A\,C\,B$, or, where necessary to avoid ambiguity, by $\angle a\,b$ or $\angle A\,C\,B$, using the symbol $\angle$ to denote an angle.

§ 85. This definition is not yet sufficient. For if the limits a and b are given the angle $a\,b$ is not uniquely determined. We may turn a half-ray from a to b either in the sense indicated by the curved arrow in fig. 16, or in the opposite sense, as in fig. 17.

FIG. 17.



We must therefore, besides the limits $a\,b$, know the sense of turning. It will be convenient to fix once for all that sense of turning which we consider as positive, and which is meant if nothing else is stated. This is best done by referring to the well-known turning of the hands of a watch.

The turning of a ray in a sense opposite to that of the hands of a watch will in future be taken as positive.

The hands of a watch, therefore, turn in the negative sense. Their motion was originally made to agree with the *apparent* motion of the sun as seen by the inhabit-

ants of the northern hemisphere. Our positive sense of turning agrees, therefore, with the sense in which the earth revolves about its axis as seen from the North Pole.

§ 86. **Angles of Continuation and of Rotation.**—If a half-ray is turned about the point C in the positive sense, from an initial position a till it coincides with the continuation b of a beyond C, the angle ab is called an *angle of continuation.* If we turn still further, the moving ray will ultimately coincide with a (fig. 19). In this case the moving ray has made a full rotation, and the angle generated is called an *angle of rotation.*

FIG. 18.

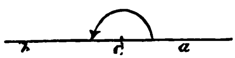
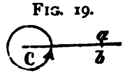
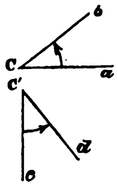


FIG. 19.



§ 87. **Equal Angles.**—Two angles are called *equal* if they can be placed in such a position that their limits, and therefore their vertices, coincide, and that both are described simultaneously by the turning of the same half-ray about their common vertex. This implies that both angles have the same sense. If the angles are of opposite sense whilst their limits coincide, they are said to be *equal in magnitude but opposite in sense.* Hence the two angles ab with vertex C and cd with vertex C', are equal if, on placing C' on C and c on a (fig. 20), the line d falls on b, and both have the same sense. If, however, after placing c on a, the moving ray arrives first at b and afterwards at d, the angle cd is said to be greater than the angle ab; or, in symbols, $cd > ab$ or $ab < cd$. (Compare § 69.)

FIG. 20.



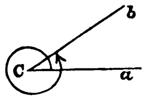
§ 88. As special cases we get:

Theorem: *All angles of continuation are equal, or equal and opposite.*

All angles of rotation are equal, or equal and opposite.

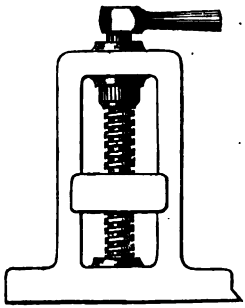
§ 89. We may turn a half-ray from an initial position *a* about *C* first through an angle of rotation, and then further on till it comes to *b*. Hence the angle *a b* is still undetermined. It may either mean the angle obtained by turning from *a* to *b*, or we may first turn through an angle of rotation and then stop at *b*, or we may first turn through two or three or any number of angles of rotation and ultimately stop at *b*. Hence, there is an indefinite number of angles which have *a* as the first and *b* as the second limit, and which are all described in the same sense. Thus *a a* may mean an angle of rotation or an angle of no magnitude, equal to zero.

FIG. 21.



In some parts of mathematics, for instance, in trigonometry or in applied mathematics, it is of importance to consider all these angles. A simple example will show this.

FIG. 22.

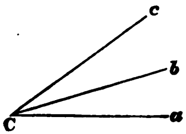


Suppose we have a common right-handed screw, arranged as in fig. 22, with an arm attached to its head, and we turn this arm in the positive sense: the nut will move downwards, and this motion of the nut will be the greater the greater the angle through which the arm has been turned. We

see at once that it is a different thing turning through an angle, say, of continuation, or through the same angle together with one, two, or more angles of rotation.

§ 90. **Adjacent Angles.**—If from a point C three rays are drawn a, b, c we have the angles ab, bc, ac. Of these any two have a limit in common. Such angles are called *adjacent angles.* Thus ab and bc, or ac and cb, or bc and ca, are adjacent angles. Further, ac is called the *sum* of the angles ab and bc; in symbols

FIG. 23.



$$ab+bc=ac.$$

This means, as in the case of a segment, that *it is the same thing whether we turn first from* a *to* b, *and then from* b *to* c, *or whether we turn at once from* a *to* c.

If c coincides with a we get

$$ab+ba=aa=\text{angle of rotation.}$$

In words: *If a ray turns first from* a *to* b, *and then in the same sense from* b *to* a, *it comes back to its original position, having completed a full rotation.*

§ 91. **Negative Angles.**—If we further agree that $-ab$ shall mean the angle which we get by turning from b to a in the negative sense, we have

$$ab+(-ab)=aa=0.$$

But if, in the last formula of § 90, we substitute for the angle of rotation aa the angle zero we have

$$ab+ba=0.$$

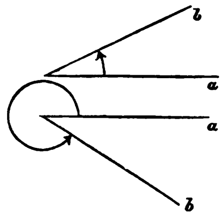
A comparison of these two formulæ shows that

In all investigations where an angle of rotation may be replaced by an angle zero we have

$$-ab = ba, \quad aa = 0.$$

§ 92. In elementary geometry it is in general sufficient to consider angles which are not greater than an angle of rotation. In future we shall, therefore, understand, if nothing else is stated, *by the angle* ab, *the angle generated by turning a ray in the positive sense, about the vertex, from* a *till it comes to* b *for the first time.* (See fig. 24.)

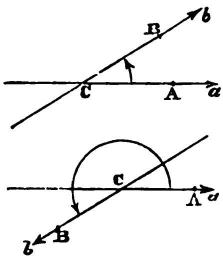
FIG. 24.



§ 93. This definition will still apply if a and b are lines unlimited in both directions, which meet each other at some point C, instead of being half-rays drawn from that point, provided we give each line a sense, which may be indicated where necessary by an arrowhead.

Definition.—*By the angle* ab *will be understood the angle included by the two half-rays* a *and* b, *which are drawn from C in the positive sense of the lines* a *and* b, or the angle ACB, as indicated in the figures. If we change the sense of *one* of the lines the angle changes its value (fig. 25).

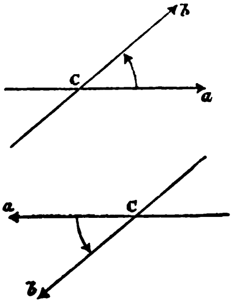
FIG. 25.



The angle which is obtained by changing the sense of *both* lines is said to be *vertically opposite* to the first angle (*fig. 26*).

Vertically opposite angles are such that the limits of the one are the continuations of the limits of the other.

FIG. 26.



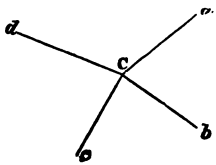
§ 94. With this definition of an angle we obtain for angles a series of formulæ exactly corresponding with those obtained in §§ 75–78 for segments, provided we always consider the angle of rotation equal to zero. But it must be kept in mind that the angle zero may, in special cases, mean a positive or negative angle of one or two or any number of rotations.

We have always

$$aa=0,\quad ab=-ba,$$
$$ab+ba=0,$$
$$ab+bc+ca=0,$$
$$ab+bc=ac,$$
$$ab+bc+\ldots+ef+fg=ag,$$
$$ab+bc+\ldots ef+fa=0.$$

Further, if the difference of two angles be defined, like the difference of segments (§ 80), as that angle which added to the second gives the first, we have

FIG. 27.



$$ab-cb=ac,$$
$$ab-ac=cb.$$

§ 95. In some cases we must retain the angle of rotation. Thus if we draw from a point C a number of half-rays, a, b, c, d, we have $ab+bc+cd+da=$ *angle of rotation.*

In words, *the sum of all the angles into which a pencil of half-rays is divided by a number of these rays equals an angle of rotation.*

Here it would, of course, be absurd to say the sum is *zero.* But whenever we have to deal merely with the relative position of the first and last limit in a sum of angles, without considering the amount of turning required to describe the angle, we may say, $aa=0$, or $aa=$angle of rotation, or $aa=$any number of angles of rotation. For either statement involves the assertion that the two limits coincide.

EXERCISES.

(1) If $AB=2$, $AC=3$, find BC.
If $AB=2$, $CA=3$, find BC.

(2) If $AB=4$, $AC=7$, $AD=-3$, $AE=2$, find BC, BD, BE, CD, CE, and DE. Verify the results by trying whether $BC+CD+DB=0$.

(3) Points A, B, C . . . on a line are often determined by their distances OA, OB . . . from a fixed point O (called the zero-point or the origin). Prove

(*a*) That always

$$AB=OB-OA.$$

(*b*) If we wish to reckon the distances of the points from a new origin Q, that

$$QA=OA-OQ.$$

(4) The positions of lines in a pencil are often fixed by the angles they make with a fixed line (called the initial line). State and prove in this case the formulæ reciprocal to those given in the last exercise about points in a row.

(5) A half-ray turns about its end point first in the positive sense through two angles of rotation, then through $\frac{1}{4}$ angle of continuation in the negative sense, through an angle of rotation in the *positive* sense, through $\frac{1}{3}$ angle of rotation in the posi-

tive sense, and lastly through $\frac{7}{8}$ of an angle of continuation in the negative sense, it being taken for granted that an angle of continuation is equal to half an angle of rotation (§ 104).

What will be the value of the angle according to the definition in § 92 between the original position of the ray and its position after each successive step of turning?

DRAWING.

We shall now add to our drawing instruments a pair of compasses and a 'scale.'

A *pair of compasses* serves to carry distances about (§ 71). It is supposed at present that none of its points is replaced by a pencil.

A *scale* serves to measure off distances in terms of a definite unit.

(6) On a line set off to scale, beginning at any point, A distances.

$$AB = +3 \text{ units of length}; \; BC = +5, \; CD = -7,$$
$$DE = -9, \; EF = +15 \text{ units of length.}$$

The positive numbers, marked +, are to be set off in the positive sense; the negative numbers, marked −, in the negative sense.

We have $AB + BC + CD + DE + EF = AF$; hence $3+5-7-9-15 = +7$.

That is, the point F ought to be, at the distance of 7 units, in the positive sense from A.

Measure AF in order to see whether your drawing is correct.

Find also, by calculation and by measuring, the distances AD, BF, CE, in order to check your calculation.

(7) Draw the figures to the first four Exercises.

CHAPTER IV.

BISECTORS OF SEGMENTS AND ANGLES.

§ 96. **Bisector of a Segment.**—A point M which divides a segment AB into two equal parts, AM and MB, so that

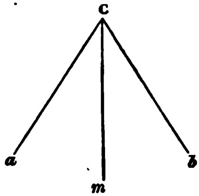
FIG. 28.

$$AM=MB, \text{ or } MA=-MB,$$

is called the *bisector*, or the *mid point* of the segment AB.

§ 97. **Bisector of an Angle.**—A line m through the vertex of an angle ab, which divides that angle into two equal parts, am and mb, so that

FIG. 29.



$$am = mb, \text{ or } ma = -mb,$$

is called the *bisector* or *mid ray* of that angle.

§ 98. The mid point M of the segment AB has the property that if the segment MB be turned about it through an angle of continuation, then B will fall on A. If the whole segment AB be thus turned about M, then the points A and B will interchange their positions.

This is expressed by saying that the two points A and B lie *symmetrically* with regard to M, and M is called a *centre of symmetry* of the figure AB.

§ 99. The mid ray m of an angle ab has the property that if the plane of the angle be doubled over by folding it along m, and turning the one angle mb about m till it falls into the plane of ma, then its limit b will coincide with a.

If the whole angle ab be turned about m till it comes back into the original plane, the two limits a and b will have interchanged their positions.

This is expressed by saying that the two lines a and b *lie symmetrically* with regard to m, and m is called an *axis of symmetry* of the figure ab.

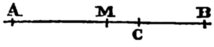
§ 100. These properties of the bisectors of segments and angles may be used practically for finding them. Thus, if a segment is given on a straight edge of a piece of paper, we may bend the paper in such a manner that the two end points coincide, and then fold down the paper. The crease formed will bisect the edge.

Similarly if a piece of paper on which we may suppose an angle given be cut along the limits of the angle, we may, by bending the paper, make the two limits coincide and fold the paper down. The crease hereby formed will be the bisector of the angle.

§ 101. These operations can, of course, not always be performed, if the figures are drawn on a solid which cannot be bent over. But farther on we shall obtain other modes of determining the bisectors of segments and angles. The operations described are, however, always conceivable.

We conclude that every segment and every angle has a bisector. Furthermore, it has only one bisector. For if we suppose a point to be moved from the mid point M of a segment AB (fig. 30) towards B, say to C, then AC will be greater than CB; hence the point C cannot bisect the segment. Similarly in the case of an angle (fig. 31). Hence

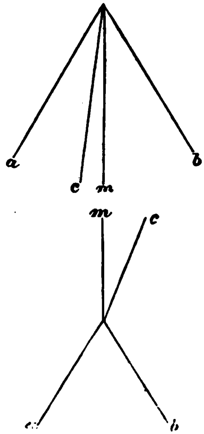
FIG. 30.



Theorem : *Every segment has one, and only one, mid point. Every angle has one, and only one, mid ray.*

§ 102. The reasoning used in the last article gives rise to the following propositions, which will be found of use hereafter:

FIG. 31.



Theorem : *The bisector of the sum of two segments or angles which have the same sense lies within the greater segment or angle (provided that the sum of the adjacent angles is less than an angle of rotation)*; and, conversely [1]

Theorem : *Of two adjacent segments or angles of like sense, that is the greater which contains the bisector of their sum.*

§ 103. The above proof for the existence of a bisector of a segment is derived from intuition. But our intuition is limited. A segment must, therefore, have both end points at a finite distance. If we suppose the point describing a segment to move to an indefinite distance, we should obtain a segment of indefinite length, and this has no bisector. For we cannot any longer conceive the segment folded over so as to make its end parts coincident.

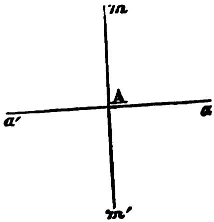
An angle, as defined in § 92, has always a bisector.

We thus see that, although there exist many points of resemblance between segments and angles, there are also important differences. A point in

[1] For the meaning of 'converse' see p. 31.

continuation beyond its vertex, this continuation m' will bisect the angle of continuation $a'a$ on the other side of the line, as is seen at once by folding over along m. We have, then, two lines intersecting at A, making all four angles equal, and therefore all four angles right angles. It follows that *all four angles contained by two unlimited straight lines are right angles if any one of them is a right angle.* The *two lines* are then said to be *perpendicular* to one another.

FIG. 33.

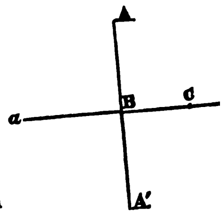


Consequently, *through a point on a line always one, and only one, line can be drawn perpendicular to the given line*, for there is one, and only one, bisector of the angle of continuation, which has its limits in the given line and its vertex at the given point.

The line perpendicular to, and passing through the mid point of, a segment is called THE PERPENDICULAR BISECTOR *of the segment.*

§ 106. Also through a point not in the line one perpendicular to the line can be drawn. To find it, we need only fold the plane over in such a manner that the crease passes through the given point, and so that the line falls on itself again. Or we may proceed thus: Fold the plane over along the given line; the

FIG. 34.



describing a segment may move to an indefinite distance, thus describing a segment of indefinite length. An angle, on the other hand, as it has been defined in § 92, is always finite. To obtain an angle of indefinite magnitude we must turn the describing line an indefinite number of times about the vertex of the angle, thus taking an unlimited number of angles of rotation.

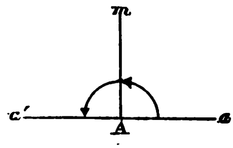
Another difference is that we have among all angles one of definite magnitude, viz. the angle of rotation, whilst there is no segment similarly distinguished from others. This gives rise to a series of propositions about angles to which no propositions about segments are reciprocal. These have now to be considered.

§ 104. **Right Angles.**—The bisector of an angle of rotation is the continuation of either of the coincident limits.

An angle of continuation is equal to half an angle of rotation.

If we bisect an angle of continuation aa' with A as vertex (fig. 32), we get two equal angles, which are called *right* angles.

FIG. 32.



A right angle is half an angle of continuation.

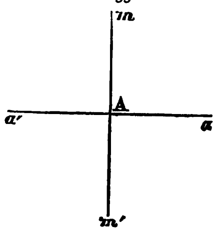
As all angles of continuation are equal, it follows.

Theorem : *All right angles are equal in magnitude.* They may of course be of opposite sense.

§ 105. **Perpendiculars.**—Two lines which include a right angle are said to be *perpendicular* to one another. If we produce the bisector m of an angle aa' of

continuation beyond its vertex, this continuation m' will bisect the angle of continuation $a'a$ on the other side of the line, as is seen at once by folding over along m. We have, then, two lines intersecting at A, making all four angles equal, and therefore all four angles right angles. It follows that *all four angles contained by two unlimited straight lines are right angles if any one of them is a right angle.* The *two lines* are then said to be *perpendicular* to one another.

FIG. 33.

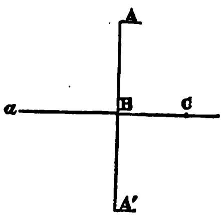


Consequently, *through a point on a line always one, and only one, line can be drawn perpendicular to the given line*, for there is one, and only one, bisector of the angle of continuation, which has its limits in the given line and its vertex at the given point.

The line perpendicular to, and passing through the mid point of, a segment is called THE PERPENDICULAR BISECTOR *of the segment.*

§ 106. Also through a point not in the line one perpendicular to the line can be drawn. To find it, we need only fold the plane over in such a manner that the crease passes through the given point, and so that the line falls on itself again. Or we may proceed thus: Fold the plane over along the given line; then the

FIG. 34.



point A will fall on some point A' on the other side of the line. Now turn back and join A to A'. The joining line cuts the given line at a point B. If we again fold the plane over, B remains where it is and A falls on A'. Hence the two adjacent angles at B, on the same side of AA', are equal, and therefore each equal to half an angle of continuation.

This can be done in *one way only*, and therefore one such perpendicular only can be drawn. For if we suppose any perpendicular, AB, drawn from A to the line a, and produce it beyond B, the angles ABC and CBA' will be equal, as the first is a right angle. Therefore on folding over along BC the line AB will fall on its production, and A on some point A', which must be the same point as before. Hence the perpendicular found before is the only one that can be drawn from A.

Theorem : *Through every point one, and only one, line can be drawn perpendicular to any given line.*

§ 107. We have thus obtained three special angles, which have a definite magnitude at whatever point in the plane we take the vertex. These are, the angle *of rotation*, the angle *of continuation*, and the *right* angle.

An angle of continuation is equal to two, and an angle of rotation to four, right angles.

Any two angles which are together equal to a right angle are said to be *complementary*, and either is called the *complement* of the other. Two angles which are together equal to an angle of continuation, or to two right angles, are said to be *supplementary*, and either is called the *supplement* of the other.

If two angles are together equal to an angle of

rotation, or equal to four right angles, the one is sometimes called the *completion* of the other.

If we denote angles by small Greek letters, and an angle of continuation by π, a right angle becomes $\frac{\pi}{2}$ and an angle of rotation 2π; consequently the above statements may be expressed as follows:

two angles α and β are *complementary* if $\alpha+\beta=\frac{\pi}{2}$,
they are *supplementary* if $\alpha+\beta=\pi$,
and one is the *completion* of the other if $\alpha+\beta=2\pi$.

From these definitions it follows that

Theorem: *Equal angles have equal complements, equal supplements, and equal completions*; and conversely

Angles which have equal complements, or supplements, or completions, are equal.

If two angles of the same sense are supplementary, each will be less than an angle of continuation or less than two right angles. Hence they may be *both* right angles, or else the one will be *greater*, the other *less*, than a right angle.

An angle is said to be acute if it is less in magnitude than a right angle, and obtuse if it is greater in magnitude than one and less than two right angles.

§ 108. If the non-coincident limits of two adjacent angles are one the continuation of the other, then the two angles are supplementary, as follows immediately from the definitions.

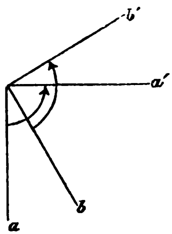
The converse also holds, viz.

Theorem: *If two adjacent angles are supplementary, the non-coincident limits are one the continuation*

of the other, or they are in a line. For the sum of the two angles equals an angle of continuation.

§ 109. If an angle ab is turned about the vertex without changing its magnitude, *then both limits describe equal angles.* For if a and b (fig. 35) are turned to a' and b', so that $ab = a'b'$, then $ab + ba' = ba' + a'b'$; hence $aa' = bb'$.

Fig. 35.



In the same manner it follows

Theorem : *If a pencil of lines be turned about its centre without changing the angles between its rays, then all lines describe equal angles.*

We may, therefore, speak of *the angle through which a figure has been turned in its plane about a point*, if we understand hereby the angle described by any ray drawn through that point.

If, in particular, an angle ab is turned through an angle of continuation, it will in the new position be vertically opposite to the original angle. Hence

Theorem : *Vertically opposite angles are equal in magnitude and sense* ; or

The two halves of a ray which turns about a point in it describe angles which are equal in magnitude and sense.

§ 110. The last proposition includes

The bisectors of vertically opposite angles are in a line.

Vertically opposite angles are symmetrical with regard to their common vertex as centre of symmetry.

Two lines of which the sense has not been fixed determine two pairs of vertically opposite angles.

Hence the bisectors of these four angles lie in two lines. Of the four given angles two adjacent ones are supplementary; their bisectors include one-half of each angle, and therefore half an angle of continuation; or

Theorem: *The bisectors of the angles formed by two intersecting lines are two lines which are at right angles to one another.*

EXERCISES.

(1) If A and B are two points in a line, determined by their distance OA and OB from a fixed origin O, and if M is the mid point of AB, prove that

$$OM = \tfrac{1}{2}(OA + OB).$$

State this result in words.

(2) If A, B and C, D are four points in a line, M and N the mid points of AB and CD respectively, then

$$MN = \tfrac{1}{2}(AC + BD) = \tfrac{1}{2}(AD + BC).$$

State this result in words.

(3) If from the ends of a segment AB segments AA' and BB' are set off which are equal in length but of opposite sense, so that $AA' = B'B$, prove that the mid point of AB is the same as the mid point of $A'B'$.

(4) If A', B', C' denote the mid points respectively of the three segments BC, CA, AB, determined by any three points A, B, C in a row, and if O denotes any other point in the row, prove that

$$OA + OB + OC = OA' + OB' + OC'.$$

(5) If A, B, C, and O are any four points in a row, if A', B' denote the mid points respectively of the segments AB and BC, and M the mid point of $A'B'$, prove that

$$2(OA' + OB') = OA + 2OB + OC.$$

(6) If A, B, C, D, and O are points in a row, A', B', C' the *mid points* respectively of the segments AB, BC, CD, if

further, A'', B'' are the mid points of $A'B'$ and $B'C'$, and if M is the mid point of $A''B''$, prove that

$$8OM = OA + 3OB + 3OC + OD.$$

(7) If a and b are two rays in a pencil determined by the angles they make with an initial line o, and if m is the bisector of the angle ab, prove that

$$om = \tfrac{1}{2}(oa + ob).$$

State this result in words, and show that it is reciprocal to that contained in exercise (1).

(8) State and prove the theorem reciprocal to that contained in exercise (2).

(9) State and prove the theorems reciprocal to those contained in exercises (3), (4), (5), and (6).

(10) From the vertex C of a given angle ab two straight lines m and p are drawn, dividing that angle respectively into two equal and into two unequal parts. Prove that half the difference between these unequal parts is equal to the angle mp, all angles being measured in the same sense.

(11) If the line p in the last question were not drawn between the limits of the given angle, would the proposition in the last exercise still hold?

Show the connection between the propositions in exercises (7) and (10).

(12) From a point four half-rays a, b, c, d are drawn. The angles ab and cd are bisected by the half-rays m and n. Prove that the angle mn is equal to half the sum of the angles ac and bd, account being taken of the sense.

Show that this is essentially the same as exercise (8).

(13) Four straight lines are drawn from a point so that of the four angles they form every two which are not adjacent are equal to one another. Prove that the four constitute, in reality, only two straight lines.

(14) Two angles, each less than an angle of continuation, are supplementary, and the greater is double of the less. Find what fraction the less is of a right angle.

DRAWING.

To bisect a segment AB, mark a point A' on it which appears to you to be about the mid point, and set off from B towards A a segment BB' equal in length to AA'. Then the mid point of AB will be the same as that of the smaller segment $A'B'$, and may be guessed at much more accurately. Assume it to be at A'', and repeat the above process with A'' instead of A' by setting off a segment BB'' equal in length to AA''. Repeat the process if necessary. The second or third trial will, as a rule, be sufficient.

Later on we shall see how to find the mid point, not by repeated trials, but by a *geometrical construction*, which would give the mid point at once with absolute accuracy, if the lines which we draw were without thickness. For practical drawing the method described is the quickest and the most accurate one.

(15) Bisect a segment by the method described.

(16) Draw the figures to the exercises (1) to (6).

(17) Draw any triangle ABC, that is a figure formed by three points A, B, C, called the vertices, and by the segments BC, CA, and AB, called the sides of the triangle. Bisect the sides and join their mid points to the opposite vertices, viz. the mid point of BC to A, and so on.

If your drawing is accurate these lines will meet in a point.

CHAPTER V.

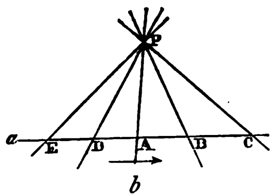
PARALLEL LINES.

§ 111. Two indefinite lines in a plane may intersect, as we have seen. We shall now consider the possibility of there being such lines which do not intersect.

In order to do this let us take, in a plane, a line *a* and a point P without it, and let us draw through

P a half-ray b which terminates at P. Let this half-ray cut the line a in A. If we now suppose this half-ray b to turn about P in the sense indicated by the arrow, the point A where it cuts the line a will move to the right along a, and will successively assume the positions B, C, . . . If we continue turning, this point will move to a greater and greater distance from its original position A, till at last it disappears to the right in the indefinite production of the line a.

Fig. 36.



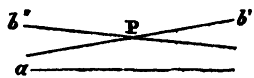
If we next turn the half-ray b from its original position PA in the opposite sense, its intersection with a will move to the left, assuming successively the positions D, E . . ., till it again disappears, but now to the left.

As soon as the point of intersection has disappeared to the right, the half-ray b, which terminates at P, does not cut the line a any longer. But perhaps its production beyond P does? Certainly if we turn far enough the production of b will meet a again.

§ 112. Let b' be the position of the half-ray b at the moment when the point of intersection with a disappears to the right, and similarly let b'' be the position of b at the moment when the point of intersection disappears to the left, and let us now produce both half-rays beyond P. We have then, instead of two half-rays terminated at P, two indefinite lines through P, and these will either coincide or include an angle.

If they include an angle, the production of b' will fall either between a and b'', and that of b'' between a and b', as in fig. 37, or these productions will lie on the opposite sides of b'' and b' respectively, as in fig. 38. If they coincide, the production of b' will coincide with b'', as in fig. 39.

Fig. 37.



§ 113. These three are the only cases conceivable. The first contradicts our axiom of the line, as we shall presently see, and is therefore inadmissible. But our axioms are not sufficient to decide which of the remaining two cases actually does occur. In looking at the figures the reader will at once feel that the third case is the true one. But this cannot be considered decisive; for the two lines may include a very small angle—that is, they may very nearly coincide without actually doing so. Or it may be that sometimes the one, sometimes the other, happens, according as we take the point P at a smaller or greater distance from the line a.

Fig. 38.

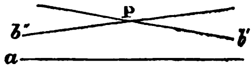
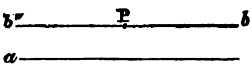


Fig. 39.



The only way of settling this point is to make an assumption, and to see whether the consequences drawn from it do or do not agree with our experience.

The assumption to be made is, that the third case only happens, and this will give us a new axiom.

Before, however, stating this axiom as a distinct proposition, let us see what the three cases really mean.

§ 114. In the first case (fig. 37) every line through

P will cut the line a; for if whilst turning round P the point of intersection disappears to the right, the production of the half-ray b will cut the line a to the left. That is, through the point P no line can be drawn which does not meet a. But every line within the angle formed by b' and the continuation of b'' would cut a at both sides, hence at two points at finite distances. This contradicts our fourth axiom (§ 45), and is therefore excluded.

In the second case (fig. 38) we should have to turn b from b' through a finite angle before the continuation would cut a again, or there would be an indefinite number of lines through P which do not cut a. But in the third case (fig. 39) there would be only *one* line b' or b'' through P which does not cut a. As soon as we turn this line about P it would meet a either to the right or to the left.

§ 115. Thus we are led to the conclusion that there exist lines in a plane which, though both be unlimited, do not meet. Such lines are called *parallel* lines, or *parallels*.

Definition of Parallel Lines.—*Two unlimited lines in a plane which do not meet are called parallel lines.*

§ 116. The assumption mentioned in § 113 may now be stated thus :—

Axiom VI. *Through a given point only one line can be drawn parallel to a given line.*

§ 117. This will hold for any line and any point in space, for we can always draw a plane through a point and a line.

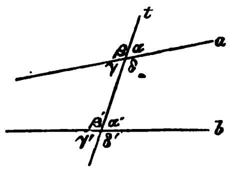
It is further to be noticed that the statement

that two lines are parallel always includes the two statements—1st, *the two lines lie in a plane*; and 2nd, *they do not meet.*

Two lines in space as a rule do not meet, as we have seen before. Two lines in a plane, however, do meet, unless they are parallel, and this is an exceptional case; for through a point we can draw an unlimited number of lines which meet a given line, and only one which is parallel to it.

§ 118. Let us next suppose that in a plane two unlimited lines a and b are given (fig. 40), and let us cut them in two distinct points by a third line t. Such a line by which a given figure is cut is called a *transversal*. At each of the points where the transversal cuts the given lines four angles are determined, $\alpha, \beta, \gamma, \delta$ with a, and α', β', γ', δ' with b. We shall consider these angles without for the moment taking account of their sense.

FIG. 40.



Four of these angles are between the given lines (viz. γ, δ, β', and α'), and are called *interior angles*; the other four (viz. α, β, γ', δ') lie outside a and b, and are called *exterior angles*. Four lie on each side of the transversal. Angles, one at each point, which lie on the same side of the transversal, the one exterior, the other interior, like α and α', δ and δ', &c., are called *corresponding angles*.

Two angles on opposite sides of the transversal, and both interior or both exterior, like α and γ', δ and β', &c., are called *alternate angles*.

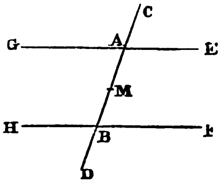
§ 119. Among these eight angles there are, at each of the points, two pairs of vertically opposite, and therefore equal, angles. If it happens that two corresponding angles are equal, for instance $\alpha=\alpha'$, then their supplements will be equal; hence $\beta=\beta'$. As $\alpha=\gamma$, $\beta=\delta$, $\alpha'=\gamma'$, $\beta'=\delta'$, these being vertically opposite angles, it follows that $\alpha=\gamma=\alpha'=\gamma'$ and $\beta=\delta=\beta'=\delta'$, and likewise that interior or exterior angles on the same side of the transversal are supplementary. For instance, γ and β' or α and δ' are supplementary. The same is true if two alternate angles are equal, for instance $\alpha=\gamma'$, or if two interior or two exterior angles on the same side of the transversal are supplementary. For if δ and α' are supplementary, then $\alpha=\alpha'$, as they have the same supplement δ.

Hence *if two corresponding or two alternate angles are equal, or if two interior or two exterior angles on the same side of the transversal are supplementary, then every angle is equal to its corresponding and to its alternate angle, and is supplementary to the angle on the same side of the transversal which is interior or exterior according as the first is interior or exterior.*

§ 120. Let us suppose that one, and therefore all, of these conditions are satisfied; then the angles on the one side of the transversal are equal to those on the other, and the figures on both sides are congruent. We may, in fact, take the figure to the right of the transversal, turn it round in the plane about the mid point M of AB (fig. 41) till A falls on B and B on A; then AE will fall on BH, as the angles EAB and HBA are equal, and BF

will fall on AG. Hence the figures coincide. If, therefore, the lines AG and BH intersect when produced, the lines AE and BF would do so too, and the lines GE, HF would have two points in common, which is impossible, according to Axiom IV. (§ 45). Consequently the two lines do not intersect, that is, they are parallel.

FIG. 41.

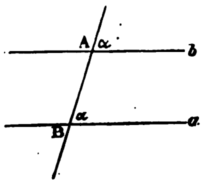


This gives the following

Theorem : *If two lines, cut by a transversal, make corresponding or alternate angles equal, or interior or exterior angles on the same side of the transversal supplementary, the lines are parallel.*

It follows, for instance, *that lines which are perpendicular to the same line are parallel.*

FIG. 42.



§ 121. The last theorem shows how we may proceed if we have to draw through a given point A a line parallel to a given line a.

We draw through A a line AB cutting a at B and making some angle, say α, with it; we then move this angle α by sliding the one limit along AB until its vertex comes to A. Its second limit will then be parallel to a, because it makes alternate angles equal. This process is practically carried out on the drawing board by aid of *set squares.*

§ 122. **Theorem** : *If two lines are parallel, then every transversal makes corresponding angles equal, alternate angles equal, and interior or exterior angles on*

the same side of the transversal supplementary. For there is only one parallel, and this may be found by making alternate angles equal.

§ 123. This may also be stated thus :

Theorem : *The necessary and sufficient condition that two lines in a plane may be parallel is that a transversal makes corresponding angles equal.* And

If corresponding angles are not equal, the two lines are not parallel; that is, they will meet if sufficiently produced.

It follows also that any two parallels have the property proved in § 120 for lines making corresponding angles equal, viz. that if we take the mid point M of any transversal AB (fig. 41), and turn the whole figure through an angle of continuation about M, the two halves of the figure on the two sides of the transversal will have interchanged places. This may be expressed, as in § 98, by saying that the figure is symmetrical with regard to M ; or

Any two parallels are symmetrical with regard to the mid point of the segment which they determine on any transversal.

§ 124. **Pencil of Parallels.**—We may have in a plane more than one line parallel to a given line. For all such lines the following propositions hold :

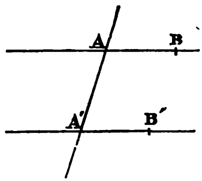
Theorem : *If two lines in a plane be parallel to a third, they are parallel to each other.* For if the two lines were not parallel they would intersect in a point, and through this point two lines would pass, both parallel to the third ; but this is against our axiom.

Further, *a series of parallel lines is cut by any other line under equal angles, or the line is equally* INCLINED *to them.*

The aggregate of all lines in a plane which are parallel to one line, and therefore parallel to each other, is called a *pencil of parallels.*

§ 125. **Sense in Parallels.**—If AB and $A'B'$ be two segments in two parallel lines, we say these are of the same sense if both lie on the same side of the line AA' (fig. 43), but they are of opposite sense if they lie on opposite sides of the line AA' (fig. 44).

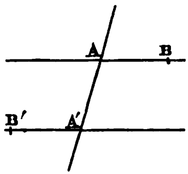
FIG. 43.



This enables us to fix the sense of one line by reference to that of a parallel line.

We need only take in each of the parallel lines a segment of the same sense as the line. The parallels are then said to be of the same or of opposite sense, according as the two segments are of the same or of opposite sense.

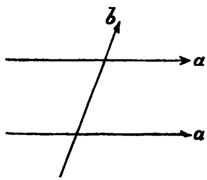
FIG. 44.



Taking account of the sense, we may now complete the theorems about angles between parallels cut by a transversal as follows (fig. 45):

Corresponding or alternate angles are equal in magnitude and sense if the parallels have the same sense; and, conversely, *if corresponding angles are equal in magnitude and sense the lines are parallel and of the same sense.*

FIG. 45.



§ 126. If we cut two parallels a and a' by two other parallels b and b', and we take one of the four angles

between ab and one of the angles between $a'b$, then these two angles are either equal or supplementary. For we may consider b as a transversal by which the two parallels a and a' have been cut. If we take in the same way one of the angles $a'b$ and one of the angles $a'b'$, then these are either equal or supplementary. Hence also any one of the angles ab, and any one of the angles $a'b'$, are either equal or supplementary. This may be stated thus :

If two angles in the same plane have the limits of the one parallel to those of the other, the two angles are either equal or supplementary.

If we take account of the sense of the lines and angles, we get at once the following

Theorem: *Two angles which have their limits parallel are equal if the limits of the one are both of the same or both of the opposite sense to the limits of the other; but the two angles are supplementary if one of the limits of the one angle is of the same, and the other of the opposite, sense to the parallel limit of the other angle.*

EXERCISES.

(1) If two lines be each perpendicular to a third, they will be parallel to one another.

(2) State and prove the converse to the above.

(3) Two lines of which one is not, whilst the other is, perpendicular to a third line will necessarily intersect.

What logical connection has this proposition with the two preceding ones?

(4) No two lines which are perpendicular respectively to two intersecting lines can be parallel to one another.

Find the logical connection between this and the above.

(5) If the limits of an angle are perpendicular to those of

another, the angles will be either equal or supplementary, according as they are of the same or of opposite sense.

(6) State the converse of exercise (5). Is it true?

State and prove the converse of (5) after having changed it to:

Of two angles of like sense the first limit of one is perpendicular to the first limit of the other; then if the second limit of the one is perpendicular to the second of the other the angles are equal. (Convert only the statement following the 'if'.)

(7) If the limits of one angle be perpendicular to those of another, then the bisectors of these angles will either be perpendicular or parallel.

(8) Investigate the converse theorem, as in exercise (6).

DRAWING.

A set square, which we now add to our drawing instruments, is commonly made of wood or ebonite in the form of a triangle which has one right angle and the other two either equal or one of them half the size of the other. The student should get one of each kind.

To draw parallel lines we place a set square with one of its edges against a straight-edge and slide it along the latter. The different positions of either of the other edges will give parallel lines (§ 121), of which we may draw as many as we like.

To draw perpendiculars the right angle of a set square is made use of. If the hypothenuse is made to slide along a straight-edge, and if in one of its positions a line be drawn along one of the limits of the right angle, and in another position a line along the other limit, then these lines will be perpendicular (§ 126).

(9) Through a given point draw a line parallel to a given line.

(10) Through a given point draw a line perpendicular to a given line, both in the case where the given point is, and where it is not, on the given line.

(11) Draw any triangle, and through its vertices draw parallels to the opposite sides. These will form another triangle.

(If your drawing is accurate the vertices of the first will be the mid points of the vertices of the second triangle.)

(12) Draw any triangle, and through its vertices draw perpendiculars to the opposite sides. (If your drawing is accurate these will meet in one point.)

(13) Draw an angle equal to the angle between two lines which meet off your drawing paper.

(14) On a line set off a number of points A, B, C . . . at equal distances. Through these points draw a series of parallel lines a, b, c . . ., and in any other direction a second series of parallel lines a', b', c' . . . Your paper will then be covered with a net of lines.

If your drawing is accurate the intersections $a'b$, $b'c$, $c'd$ . . . will all lie in a line. So will the points $a'c$, $b'd$ . . . Draw these lines and all others of a similar kind which you can discover.

CHAPTER VI.

ANGLES IN POLYGONS.

§ 127. **Broken Lines.**—If any number of points A, B, C, D... be given in a plane, and these be joined in any order, the first to the second, the second to the third, and so on, by segments of lines which terminate at these points, we obtain what is called a *broken line*. The first and last of the given points are its *end points*.

This broken line may be supposed described by a point moving from A to B along their join, then from B to C, and so on.

The segments AB, BC, CD . . . are called the *sides* of the broken line. Each side, as well as the *whole broken* line, has a *sense*.

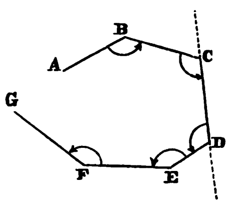
At each of the given points, with the exception of the end points, two sides meet, one having there its end point and the other its origin.

These points are called the *vertices* of the broken line.

§ 128. *The number of sides is one less than the number of given points*, for every one of the points, excepting the last, is the origin of one side.

The number of vertices is two less than the number of given points, for every one, with the exception of the end points, is a vertex.

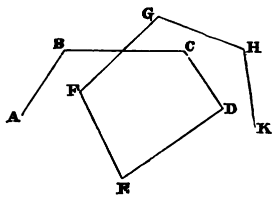
FIG. 46.



§ 129. **Angles of a Broken Line.**—At each of the vertices two sides meet and determine two angles, of which one is less, the other greater, than an angle of continuation.

If we turn the first side AB about B in a *given sense* till it coincides with the next side BC, it describes one of the angles at B. If we next turn BC about C in the same sense to CD, then CD about D to DE, and so on, always turning in the same sense, we get at each vertex one angle. Every side is, then, the limit of two angles which lie on the same side of it.

FIG. 47.

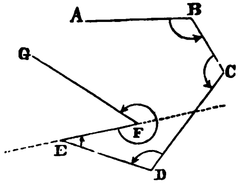


The angles thus determined are called the angles of the broken line *taken in the given sense.*

If we change the sense of turning, every angle changes into its completion.

§ 130. It may happen that in one of these two sets of angles every one is less than an angle of continuation. In this case the broken line has the following property: if one side be produced both ways, then the two adjacent sides lie on the same side of it. The broken line is then said to be *convex*, and in this case those angles only are called the angles of the broken line which are each less than an angle of continuation.

FIG. 48.

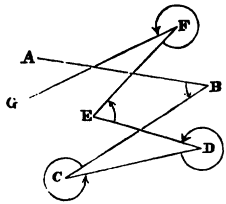


Thus the figures 46 and 47 show convex broken lines, whilst the broken line in fig. 48 is not convex, for the sides DE and FG are on opposite sides of the side EF.

The broken line in fig. 49 also is not convex.

§ 131. **Exterior Angles.**—If in a broken line every side be produced in the sense in which a moving point describes the broken line, we obtain at each vertex an angle between the one side produced and the next following side. This angle is called an *exterior angle*.

FIG. 49.



We thus have at every vertex an exterior angle. These are all to be taken in the same sense.

§ 132. **Polygons.**—If in a broken line the two end points coincide, the figure obtained is called a *polygon* (*figs. 50 and 53*), and the broken line its *perimeter*.

A polygon has as many vertices and angles as it has sides. The angles are determined as in the case of the broken line.

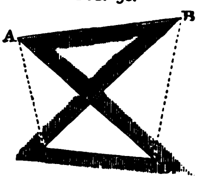
A polygon has a *sense*. According as the perimeter is described in the one or in the opposite sense, we say that the polygon itself is of the one or the opposite sense.

§ 133. If no side of a polygon cuts another it bounds one finite connected part of the plane, which is called the *area* of the polygon, and we may speak of points *within* and *without* the polygon.

In this case we understand by the angles of the polygon those angles of which the part near the vertex lies within the polygon, these angles being all taken in the same sense. If each of these angles is less than an angle of continuation, the polygon is said to be convex. If one or more angles are greater than an angle of continuation, the polygon is said to have one or more *re-entrant* angles, provided that no two sides of the polygon intersect.

§ 134. If the polygon cuts itself once or several times, then not a single part of the plane is bounded by it, but two or more disconnected portions.

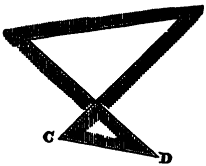
FIG. 50.



In this case it depends, as in the case of the broken line, upon the sense in which the angles are taken, which of the two angles between two consecutive sides we take as the angle of the polygon. Thus in the polygons (figs. 50 and 51) we may take *either* the angles shaded in the first, or those

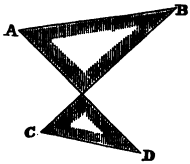
shaded in the second, as the angles in the polygon; but it would be against our definition to consider those shaded in fig. 52 as the angles in the polygon, for the side BC or DA would have the two angles adjacent to it on opposite sides of it.

FIG. 51.



§ 135. A polygon has at least three sides. In this case it is called a *triangle*, and is always convex.

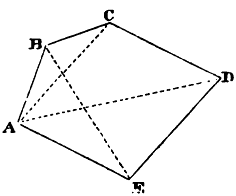
FIG. 52.



Polygons with 4, 5, 6, 8, 10, 12 sides are respectively called *tetragon* or *quadrilateral*, *pentagon*, *hexagon*, *octagon*, *decagon*, *duodecagon*.

A polygon which has all its angles equal, and also all its sides equal, is said to be *regular*. A regular polygon is either convex or its sides cut each other, but it cannot have re-entrant angles. A regular polygon whose sides cut each other is also called a *star polygon*.

FIG. 53.

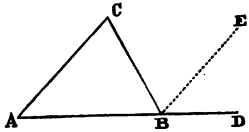


§ 136. **Diagonals.**—Any line joining two vertices is a side if the two vertices are consecutive ones, otherwise it is called a *diagonal*. Thus in fig. 53 the lines AC, AD, &c., are diagonals; whilst in fig. 50 the line AC or BD is a diagonal.

§ 137. Let us consider the simplest of all polygons

the triangle, and in a triangle ABC let us produce (fig. 54) a side AB to D; then the angle CBD contained by the side CB and the side AB produced will be equal to the sum of the two angles at C and A in the triangle. For if through B a line BE be drawn parallel to AC, it will divide the angle CBD into two parts. Of these angle EBD is equal to CAB, they being corresponding angles, and angle CBE is equal to ACB, they being alternate angles. But the angles ABC, CBE, and EBD form together an angle of continuation, and as they are equal respectively to the three angles in the triangle, we have

FIG. 54.



Theorem: *The sum of the angles in any triangle is equal to an angle of continuation*, or *equal to two right angles*, or $=\pi$.

§ 138. At the same time it has been proved

Theorem: *In every triangle an exterior angle is equal to the sum of the two interior and opposite angles*; or

Every angle in a triangle is supplementary to the sum of the other two.

This shows also that in a triangle at least two angles are acute. The third angle may be *acute*, or *right*, or *obtuse*; and the triangle is called *acute-angled*, *right-angled*, or *obtuse-angled* accordingly.

In a right-angled triangle the side opposite the right angle is called the *Hypothenuse*.

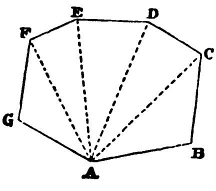
§ 139. These are very important theorems. They are immediate consequences of the axiom about

parallels. If that axiom be not the true one, we ought to have made in § 113 the other possible assumption. But it can be proved that the sum of the angles in a triangle would in that case be less than two right angles, by a quantity which increases with the size of the triangle.

Angles, however, can be measured with very great accuracy, and in many triangles the angles have been measured. In every case, even with triangles having sides many miles long, the sum of the angles has been found to be equal to an angle of continuation, at least so nearly that the difference may be accounted for by inaccuracy in measurement. If we ought to have made in § 113 the other assumption, this sum should have been found different from an angle of continuation. Experience thus confirms our axiom.

§ 140. Every convex polygon may be divided by diagonals into triangles. The simplest way of doing this is by drawing all diagonals which pass through one vertex A. If we leave out the two sides adjacent to A, in the figure the two sides AB and AG, every other side of the polygon determines one triangle with a vertex at A. The number of these triangles is thus two less than the number of sides. Hence a convex polygon of n sides can be divided into n-2 triangles, such that their angles together make up the angles in the polygon. But the sum of the angles in each triangle is equal to an angle

Fig. 55.



of continuation. Therefore the sum of all the angles in the convex polygon is equal to n-2 angles of continuation, or equal to 2(n-2) right angles. Hence

Theorem: *The sum of all the angles in a convex polygon is equal to as many angles of continuation less two as the polygon has sides*; or,

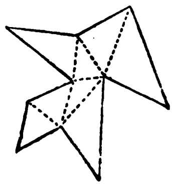
In a convex polygon of n *sides the sum of all the angles equals* 2n-4 *right angles.*

Thus the sum of the angles in a convex quadrilateral is equal to four right angles.

§ 141. **Sum of Exterior Angles.**—If all the sides of a convex polygon be produced in the same sense, we get an exterior angle at every vertex. This is supplementary to the adjacent angle in the polygon. In a convex polygon of n sides we have, therefore, n exterior angles, which, together with the adjacent interior angles, form n angles of continuation. But the interior angles are together equal to n-2 angles of continuation; therefore

Theorem: *In any convex polygon the sum of the exterior angles* (one at each vertex) *equals two angles of continuation, or four right angles.*

FIG. 56.



§ 142. These two theorems about the angles in a polygon have been proved for convex polygons only. The first is also true for polygons with re-entrant angles, provided that no side cuts another. The proof is very similar. It is always possible to divide the polygon by diagonals into n-2 triangles (fig. 56). The complete proof will be left to the student. The second theorem

about the exterior angles, however, does not hold in the above form for other than convex polygons.

EXERCISES.

(1) In every right-angled triangle the two acute angles are complementary.

(2) If one angle of a triangle be equal to the sum of the other two, what is its magnitude?

(3) State and prove the converse to the proposition in exercise (1), and show that the answer to exercise (2) is a logical consequence of (1) combined with the theorem in § 137.

(4) Find the value of the sum of the angles in a convex polygon of four, five, six, seven, and eight sides.

(5) Find the value of an angle in a regular convex polygon of three, four, six, eight, and twelve sides.

(6) Prove that a convex polygon cannot have more than three obtuse exterior angles, and not more than three acute interior angles.

(7) The bisectors of two angles in a triangle, produced till they meet, include an obtuse angle.

(8) If the sides of one triangle be respectively perpendicular to those of another, the angles of the one will be respectively equal to those of the other.

(9) Determine the magnitude of the acute angles in each of the two set squares described in Drawing Exercises to Chapter V.

(10) Through a point on a line, and on the same side of the line, four half-rays are drawn, one perpendicular to the given line and the others making angles with it equal to the acute angles of the set squares mentioned in the last exercise. Determine the values of the angles contained by these rays.

DRAWING.

The first seven of the following problems should be solved by aid of the set squares only.

(11) Draw the figures to exercises (8) and (10).

(12) Divide a right angle into two, into three, and into six *equal parts.*

(13) Draw angles which have respectively the magnitude of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, and $\frac{5}{6}$ of a right angle.

(14) Draw triangles which have their angles equal respectively to the angles in the set squares.

(15) Draw a triangle which has all its angles equal. [See exercise (5).]

(16) Draw a quadrilateral and also a hexagon which have all their angles equal.

(17) Through a given point in a given line draw two half-rays so that the angle between them is bisected by the given line.

This may be done in eleven different ways, according as the angle between the given line and one of the half-rays equals 1, 2, 3 . . . 10, or 11 times $\frac{1}{6}$ of a right angle.

(18) Draw a convex polygon which has three of its interior angles acute.

(19) Draw a polygon which has three of its exterior angles obtuse.

CHAPTER VII.

AXIAL AND CENTRAL SYMMETRY.

§ 143. IN the previous chapters the fundamental notions of geometry and the elementary properties of the simpler figures have been developed. In particular we have obtained a series of axioms, that is, of propositions taken from experience. To these no new ones will be added. Those already given are necessary and sufficient to characterise space as we conceive it. In what follows we shall develop further consequences of these axioms, and of the propositions already derived from them relating to segments and angles.

It will now be our task to investigate properties of

more complicated figures. These investigations will be limited to properties of figures which are congruent and to consequences flowing from these properties.

§ 144. Congruent figures have been defined (§ 39) as figures which, when applied to one another, can be made to coincide. In other words, two figures are congruent if it is possible to place the one on the other in such a manner that every point in either falls on, and coincides with, some point in the other. The one figure is, in fact, an exact copy of the other. Both have the same shape and the same size, but they differ in position, and position only.

To a plane figure, a plane figure only can be congruent. Their planes may be distinct, and may lie anywhere in space, or else their planes may lie one on the other. In the latter case, which is the one with which we are most concerned, the two figures occupy different positions in the same plane. But even then we shall suppose that each figure lies in its own distinct plane, that it is possible to separate these and to move either figure with its plane into any position that may appear convenient for our investigations. Of such positions two are of special interest.

§ 145. **Corresponding Points.**—Let us suppose that the two figures coincide. In this position every point A in the one coincides with a point A' in the other. These points will be said to *correspond* the one to the other. Hence

To every point in one of two congruent figures, there corresponds one, and only one, point in the other, those points being called 'corresponding' which coincide if the two figures are applied to one another.

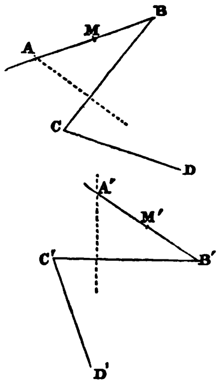
Similarly we have to every line in the one a corresponding line in the other, to every segment or angle in the one a corresponding segment or angle in the other, and always to every part of the one figure a corresponding part of the other. Hence

Definition: *In two congruent figures those parts are called corresponding which coincide if the whole figures are made to coincide.*

It follows that—*Corresponding parts of congruent figures are themselves congruent.*

§ 146. This implies: if we know that two figures are congruent, and if we further know that to any two points A and B (fig. 57) in the one there correspond the two points A' and B' in the other, then we know also that the line through A' and B' corresponds to the line through A and B, and that the segments AB and $A'B'$ are equal in length. If, besides, C and C' are corresponding points, then we know that to the angle ABC there corresponds the angle $A'B'C'$, and that these angles are equal; for if we apply the one figure to the other these corresponding points, segments, and angles will coincide.

FIG. 57.



In the same manner it will be seen that to the mid point M of the segment AB corresponds the mid point M' of the corresponding segment $A'B'$.

If we draw in the one figure from A a perpendicular

to the line BC, and in the other figure from the corresponding point A' a perpendicular to the line $B'C'$ corresponding to BC, then these perpendiculars will be corresponding lines. For if we apply the figures again, A' falls on A, $B'C'$ on BC, and the perpendiculars must coincide, as only one perpendicular can be drawn from a point to a line.

Thus we arrive at propositions like the following:

To the join of two points (or lines) in the one figure corresponds the join of the two corresponding points (or lines) in the other.

The distance between two points in the one figure is equal to the distance between the two corresponding points in the other.

The angle between two lines in the one is equal to the angle between the two corresponding lines in the other.

§ 147. **Symmetry with regard to an Axis.**—All these properties are true for every position of the two figures. But if we wish to see which points and lines are corresponding we must apply the one figure to the other. This operation, however, can only be performed as a mental conception. We must therefore find other means for comparing the two figures. The investigation of the properties required for such comparison is greatly simplified by placing the two figures in convenient positions, and we shall see that two positions are particularly useful for this purpose.

To obtain these, let us start with the position of coincidence, and let us take in the common plane *any line* s (fig. 58). We may then turn the plane of *the one figure* about this line s till its plane, after half

a revolution, coincides again with the plane of the other figure. The two figures themselves will then have distinct positions in the same plane. But they will have this property, that they can be made to coincide by turning the one figure about the line s, that is, by folding the plane over along that line.

Two figures in the same plane which have this property are said to be *symmetrical with regard to the line* s *as axis of symmetry.*

An example of such symmetry we have had in § 99, where it was shown that two intersecting lines are symmetrical with regard to a bisector of their angle. As a special case of this, or as a consequence of the reasoning used in § 105, we see that a perpendicular to a line a is symmetrical to its continuation with regard to the line a as axis. In fig. 34 (§ 106) the two points A and A' are symmetrical with regard to the line a as axis.

§ 148. **Symmetry with regard to a Centre.**— If, on the other hand, we take in the common plane of the two coincident figures any point S, instead of a line s, we may turn the one figure about this point so that its plane slides, whilst turning, over the other plane without ever separating from it.

Let this turning be continued till one line through S, and therefore (§ 109) the whole figure, has been turned through an angle of continuation about S.

The two congruent figures still lie in the same plane, and have such positions that the one can be made to coincide with the other by turning it in the plane through an angle of continuation about the *fixed point* S *(fig. 59).*

Two figures which have this property are said to be *symmetrical with regard to the point S as centre of symmetry.*

Examples of central symmetry have already occurred. Two points are symmetrical with regard to the mid point of the segment joining them (§ 98). Two vertically opposite angles are symmetrical with regard to their common vertex (§ 110), and from the reasoning used in § 120 it follows that two parallel lines are symmetrical with regard to the mid point of any segment having one end-point in each.

§ 149. We have thus obtained two kinds of symmetry. The one is symmetry with regard to an axis, called, for shortness, *axial symmetry.* The other is symmetry with regard to a centre, or *central symmetry.*

These two kinds of symmetry stand in the relation of reciprocity which has been explained in § 62. For where we took a line in the one case we took a point in the other, about which the planes containing the two figures were turned.

It is therefore to be expected that both have a great many reciprocal properties. These will be brought prominently forward in §§ 151–153.

§ 150. Two figures which are symmetrical are by definition congruent. But it is also true that *any two congruent plane figures can always be placed, and this in an infinite number of different ways, in such positions that they are symmetrical with regard to an axis or with regard to a centre.* To do this we need only apply the one figure to the other so that they *coincide,* then select in their common plane any line

as axis, or *any* point as centre, and turn the one plane about this line or point, as before described.

The statement that two figures are symmetrical implies, therefore, *first*, that they are congruent, and *secondly*, that they lie in particular relative positions.

Some of the properties following from the first condition have already been stated. Everything relating to the correspondence of parts in congruent figures holds for symmetrical figures. These properties will be shortly repeated, and then the additional properties peculiar to the position of symmetry will be studied.

The student should not omit to go through all the exercises in drawing at the end of this chapter; for the special cases given there, and more particularly the actual drawing of the figures, will greatly facilitate the understanding of the general propositions laid down in the next paragraphs.

The latter should be read again when these exercises have been gone through.

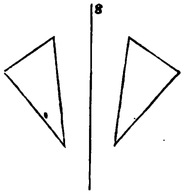
As the two kinds of symmetry are reciprocal, it follows that to a property relating to lines in the one case we must have a property relating to points in the other, and *vice versâ* (§ 62). And similarly for a property in either relating to segments we get in the other a property relating to angles.

It is customary to print reciprocal theorems and reciprocal investigations on opposite halves of a page, broken into two columns, and this will be done in the following paragraphs.

§ 151*a*. **Axial Symmetry.**—***Definition***: *If*

§ 151*b*. **Central Symmetry.**—***Definition***: *If*

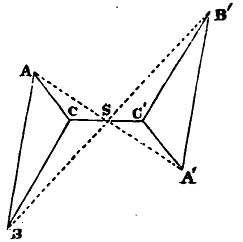
FIG. 58.



two figures in the same plane can be made to coincide by turning the one about a fixed line in the plane through an angle of continuation, the two figures are said to be symmetrical with regard to that line as AXIS OF SYMMETRY. *If the two figures are halves of one figure the whole figure is said to be symmetrical with regard to the axis, and this axis is said to be an* AXIS OF SYMMETRY, *or simply an* AXIS *of the figure.*

§ 152*a*. *To a point, or line, or angle, &c., in the one figure corresponds a point, or line, or angle, &c., in the other figure.*

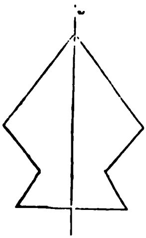
FIG. 59.



two figures in the same plane can be made to coincide by turning the one about a fixed point in that plane through an angle of continuation, the two figures are said to be symmetrical with regard to that point as CENTRE OF SYMMETRY. *If the two figures are halves of one figure, the whole figure is said to be symmetrical with regard to a centre, and this centre is said to be a* CENTRE OF SYMMETRY, *or simply a* CENTRE *of the figure.*

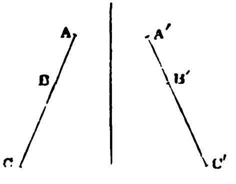
§ 152*b*. *To a line, or point, or segment, &c., in the one figure corresponds a line, or point, or segment, &c., in the other figure.*

FIG. 60.



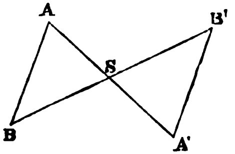
To the join of two lines corresponds the join of the corresponding lines, to the join of two points the join of the corresponding points, to the segment between two points the segment between the corresponding points, to the angle between two lines the angle between the corresponding lines, and so on.

FIG. 62.



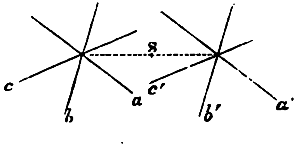
To three or more points in a line correspond three or more points in the corresponding line ; or

FIG. 61.



To the join of two points corresponds the join of the corresponding points, to the join of two lines the join of the corresponding lines, to the angle between two lines the angle between the corresponding lines, to the segment between two points the segment between the corresponding points, and so on.

FIG. 63.

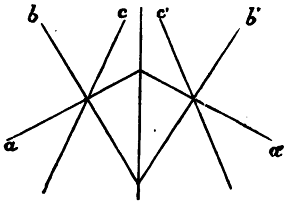


To three or more lines in a point correspond three or more lines in the corresponding point ; or

If three points A, B, C (fig. 62) *lie in a line, the three corresponding points A′, B′, C′ lie in a line which corresponds to the first.*

To three or more lines in a point correspond three or more lines in the corresponding point; or—*If three lines* a b c *lie in a point* (fig. 64), *the three corresponding lines* a′ b′ c′ *lie in a point which corresponds to the former.*

FIG. 64.



Corresponding segments are equal in length.

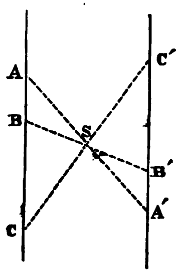
As they are not necessarily parallel, we cannot compare their sense.

Corresponding angles are equal and of opposite sense.

If three lines a, b, c (fig. 63) *meet in a point, the three corresponding lines* a′, b′, c′ *meet in a point which corresponds to the former.*

To three or more points in a line correspond three or more points in the corresponding line; or—*If three points A B C* (fig. 65) *lie in a line, the three corresponding points A′ B′ C′ lie in a line which corresponds to the former.*

FIG. 65.



Corresponding angles are equal in magnitude and sense.

Corresponding segments are equal and of opposite sense.

To parallel lines correspond parallel lines.

To a perpendicular corresponds a perpendicular.

To the bisector of an angle or segment corresponds the bisector of the corresponding angle or segment.

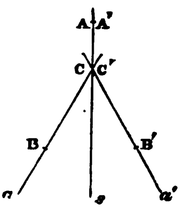
To equal segments or angles correspond equal segments or angles.

Corresponding polygons are congruent but of opposite sense (§ 132).

§ 153*a*. *Every point in the axis corresponds to itself.*

The axis corresponds to itself.

FIG. 66.



Every point which corresponds to itself lies in the axis.

To parallel lines correspond parallel lines.

To a perpendicular corresponds a perpendicular.

To the bisector of a segment or an angle corresponds the bisector of the corresponding segment or angle.

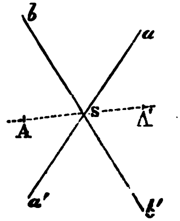
To equal angles or segments correspond equal angles or segments.

Corresponding polygons are congruent and of like sense (§ 132).

§ 153*b*. *Every line through the centre corresponds to itself.*

The centre corresponds to itself.

FIG. 67.



Every line which corresponds to itself lies in the centre (*passes* through it).

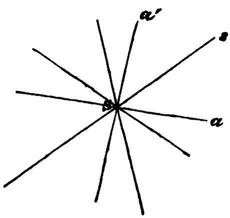
The join of two corresponding lines lies on the axis, for it corresponds to itself.	*The join of two corresponding points lies in the centre,* for it corresponds to itself.
Two corresponding lines are equally inclined to the axis; their join lies on the axis; or— *The angle between two corresponding lines is bisected by the axis.*	*Two corresponding points are equidistant from the centre; their join passes through the centre*; or— *The segment between corresponding points is bisected by the centre.*
The join of two corresponding points is perpendicular to the axis, and the segments between the points are bisected by it.	*The join of two corresponding lines does not exist*; THEY ARE PARALLEL, for they make equal angles with every line through the centre.
Corresponding points are equidistant from the axis.	*Corresponding lines are equidistant from the centre*; that is, the perpendiculars to them from the centre are equal in length.
Every line perpendicular to the axis corresponds to itself, and cuts corresponding lines in corresponding points.	*Every line through the centre cuts corresponding lines in corresponding points.*
Points which join two pairs of corresponding lines are equidistant from the axis, and have their join bisected perpendicularly by the axis.	*Lines which join two pairs of corresponding points are parallel.*

To a proposition relating to lines or angles in axial symmetry we have a proposition relating to points or segments in central symmetry.

To a proposition relating to points or segments in central symmetry we have a proposition relating to lines or angles in axial symmetry.

§ 154. **Examples of Symmetry.**—The present and the following paragraphs contain important examples of symmetry in simple figures.

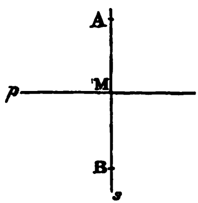
FIG. 68.



If a figure consists of a single point or of a pencil of lines (fig. 68), then this point may be considered as a centre of symmetry, and any line s through it as an axis of symmetry. In the latter case any two lines a, a' equally inclined to the one chosen as axis are corresponding lines.

If a figure consists of two points A, B (or pencils) (fig. 69), then their mid point M is a centre of symmetry, and the two points correspond one to the other.

FIG. 69.



The join s of the two points, on the other hand, is an axis, each of the given points now corresponding to itself. The line perpendicular to the join and passing through the mid point is a second axis, the two given points again corresponding the one to the other. This line is

called the *perpendicular bisector* of the segment AB determined by the two points.

§ 155. If a figure consists of a *single line*, or *row of points*, then this line may be considered as *axis* of symmetry and every point in it as *centre*. In the latter case any two points equidistant from the one chosen as centre are corresponding points. Further, every perpendicular to the given line is an *axis* of symmetry.

If a figure consists of *two intersecting lines*, then each of the two bisectors of the angles formed by the two lines is an *axis* of symmetry. These lines are *at right angles* to each other (§ 110). With regard to each of them as axis, the two given lines correspond *the one to the other*.

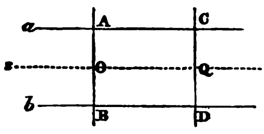
If the two given lines are at right angles, not only the bisectors of their angles, but either of the given lines, may be taken as axis. In this case each line corresponds to *itself*.

The join of any two intersecting lines is a *centre* of symmetry, every line corresponding to itself.

In all these cases the reader should make figures for himself.

§ 156. If the two lines a and b are parallel they have common perpendiculars. Every one of these perpendiculars is an axis of symmetry, for it is an axis for each of the given lines. If one of these perpendiculars cuts the given lines at A and B (fig. 70), then *the mid point O of AB is a centre* of symmetry (§ 123). *Hence the figure has an infinite number of centres.*

FIG. 70.



If $O Q$ be the perpendicular bisector of $A B$, then it is parallel to the given lines a and b, as all three are perpendicular to $A B$. If we fold the plane over along this line $O Q$, then B will fall on A and $B D$ on $A C$, as both lines are perpendicular to $O A$. Hence the line $O Q$ is an axis of symmetry. As this line divides the strip of the plane between the two parallels a and b into two congruent parts, it will be called the *bisector of the strip between the parallels.* Accordingly two parallel lines have the *bisector of the strip* between them, and also *every perpendicular to it*, an *axis* of symmetry, whilst every point in the bisector of the strip is a *centre* of symmetry.

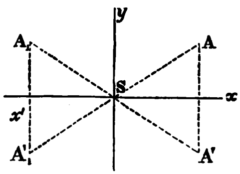
§ 157. A figure consisting of a *line and a point* without it has no *centre* of symmetry, for a point which shall be a centre of the line must lie on the line, whilst the given point has only itself as a centre. But the figure has an *axis*. Every line through the point is an axis of the point, and every line perpendicular to the line is an axis of the line; hence the perpendicular from the point to the line is an axis for each, and therefore for the figure.

§ 158. **Two Axes of Symmetry.**—In several of these figures we have found two axes of symmetry at right angles to one another. Thus two points or two intersecting lines have two axes perpendicular to one another. In each case the join of these axes is a centre. In the case of two parallels we found an infinite number of parallel axes, and another axis perpendicular to them. Here, again, every join of two of these axes is a centre. It is easily proved that this property always holds, whence we have the following theorem:

Theorem: *If a figure has two axes of symmetry at right angles to one another, then the join of these axes is a centre of symmetry.*

For if x and y be two axes at right angles (fig. 71), then to a point A will correspond a point A' with regard to x as axis. To these will correspond points A_1 and A_1' with regard to y as axis. These points A_1 and A_1' will correspond to each other with regard to x. To see this, let us first fold over along y; then A falls on A_1 and A' on A_1'. If we now, without folding back, fold over along x, A' and with it A_1' will fall on A, which coincides with A_1. At the same time SA and SA_1' coincide, so that the angles ASx and $A_1'Sx'$ are equal, where x' denotes the continuation of x beyond S. It follows that ASA_1' are in a line, and that the segment AA_1' is bisected at S; or S is a centre of symmetry for AA_1', and similarly for A_1 and A'.

FIG. 71.



EXERCISES.

In axial symmetry prove that:

(1) Lines joining any point on the axis to two corresponding points are corresponding lines.

(2) Every line perpendicular to the axis cuts corresponding lines in corresponding points.

(3) Two lines which pass

In central symmetry prove that:

(1′) Points in which any line through the centre cuts corresponding lines are corresponding points.

(2′) Parallels through corresponding points are corresponding lines.

(4) Two lines which pass

through two corresponding points A, A', and make equal but opposite angles with AA', are corresponding lines.	through two corresponding points A, A', and make equal angles with AA', are corresponding lines.

(5) In axial or central symmetry if A, A' and B, B' are pairs of corresponding points, the lines AB and $A'B'$, as well as AB' and $A'B$, are corresponding lines; and if a, a' and b, b' are pairs of corresponding lines, the points ab and $a'b'$, as well as ab' and $a'b$, are corresponding points.

(6) Perpendiculars to the limits of an angle meeting these at equal distances from the vertex intersect on the bisector of the angle.

(7) If on two parallels through the ends of a segment AB two points C, D be taken such that $AC = DB$, the line CD will bisect AB.

(8) Two equal and opposite segments AB and $A'B'$ lie on parallel lines. Determine a point S such that A and A', as well as B and B', are symmetrical with regard to S as centre.

(9) If two triangles ABC and $A'B'C'$ have the sides of the one parallel respectively to those of the other, but of opposite sense, if further one side in one is equal to the parallel side in the other, then the three lines which join corresponding vertices, viz. those which lie opposite to parallel sides, meet in a point, which is the mid point of each.

(10) State and prove an analogous theorem about axial symmetry.

DRAWING.

In the following constructions of symmetrical figures, it is supposed that either the axis or the centre of symmetry are given.

(11) To a given point A find the corresponding point A'.

Solution in Axial Symmetry.	*Solution in Central Symmetry.*
In the perpendicular drawn from A to the axis and cutting the latter at M determine A' so that $AM = MA'$.	The point A' lies on the line joining A to the centre S, making $AS = SA'$.

(12) To any given line a find the corresponding line a'.

Solution in Axial Symmetry.

To any two points A, B in a determine the corresponding points A', B'. Their join gives a'.

When possible take B where a cuts the axis.

If a is parallel to the axis draw a' parallel to it through A'.

Solution in Central Symmetry.

Take any point A in a and determine its corresponding point A'. The line through A' parallel to a will be the required line a'. (§ 153, *b.*)

(13) By aid of the set squares draw a pair of lines such that a given line may bisect one of the angles included between them (or so that the given line may be an axis of symmetry).

Solution is possible in eleven different ways. [Chap. VI. Ex. (17.)]

(14) To a given point A find the corresponding point A' by aid of set squares only.

Solution in Axial Symmetry.

Draw a pair of corresponding lines l, l' [exercise (13)] of which l passes through the point A. Then the point A' will be the point where the line l' cuts the perpendicular from A to the axis.

Solution in Central Symmetry.

Join A to the centre S and draw a line p perpendicular to $A\,S$ through S. Find (as in opposite column) the point A' corresponding to A with regard to p as axis. This will be the required point.

(15) Having given the axis or centre of symmetry and one pair K, K' of corresponding points, construct a figure symmetrical to a given figure by aid of a straight-edge only, using neither a pair of compasses nor a set square.

Solution in Axial Symmetry.

A being any point in the figure, draw $A\,K$ and $A\,K'$ *cutting* the axis in V and V'. *The lines joining* these two

Solution in Central Symmetry.

A being any point in the figure, draw through K' a line parallel to $A\,K$. This cuts the line $A\,S$ at A'. Or: Through

points crosswise to K and K', viz. KV' and $K'V$, meet in the point A', corresponding to A.

K' draw a line parallel to AK, and through K a line parallel to AK'. They meet in A'.

To test the accuracy of your drawing see whether the axis or the centre bisects the segment AA'.

(16) To a given triangle ABC construct the corresponding triangle $A'B'C'$, both in axial and central symmetry.

First by method in exercise (11).

Second by finding A' as before and the rest by the method in exercise (15).

Check each construction by the other.

(17) To a four-point find the corresponding four-point with regard to a given axis or a given centre.

Determine in each figure the triangle formed by the three joins of opposite sides.

Check your construction.

(18) Draw a figure symmetrical to a triangle with regard to one of its sides as axis.

There are three cases, according as each of the angles adjacent to the side chosen as axis is acute, or one is acute and the other right or obtuse.

(19) Draw the figure symmetrical to a triangle with regard to the perpendicular bisector of one of its sides as axis.

(20) Draw the figure symmetrical to a triangle with regard to one of its vertices, and also with regard to the mid point of one of its sides, as centre.

(21) Draw the figures symmetrical to a triangle with regard to the mid point of each side.

(22) Having given two corresponding points A and A', find the axis of symmetry by aid of set squares.

Solution: Through A and A' draw two lines making equal but opposite angles with AA'. The perpendicular from their join to AA' will be the axis required.

(23) Bisect a given segment by aid of the set squares.

(24) Find the centre of symmetry, having given two corresponding points.

(25) Find an axis of symmetry so that two given lines a and a' correspond to each other. (This cannot be done by aid of set squares only.)

Solution : From the join V of the given lines set off equal segments VA on a and VA' on a'. The perpendicular from V to AA' is an axis of symmetry. The parallel to AA' through V is another axis.

(26) Determine the bisector of any given angle, and also the two bisectors of the angles made by two intersecting lines.

(27) Draw the bisector of an angle included by two lines which meet off the drawing paper.

What becomes of the bisector in this construction if the lines are parallel?

(28) Find the centre of symmetry, having given two pairs of corresponding (hence parallel) lines.

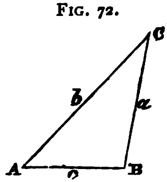
(29) Draw the figures mentioned in §§ 154 to 157.

CHAPTER VIII.

THE TRIANGLE.

§ 159. **The Triangle.**—A triangle has three vertices and three sides. Each side joins two vertices and is *opposite* the third, whilst each vertex joins two sides and is *opposite* the third. Hence every side has a vertex opposite, and every vertex has a side opposite.

FIG. 72.



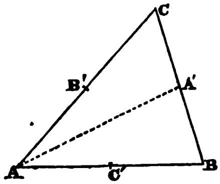
It will be convenient to denote the vertices by capital letters A, B, C, and the sides by small letters a, b, c, in such a manner that a vertex and *the opposite* side are denoted by the same letter.

Hence a is the side opposite A, and so on, as in the figure (fig. 72); or

$$BC=a, \ CA=b, \ AB=c.$$

Every side has a mid point. These will be denoted by A', B', C', so that A' is the mid point of the side opposite A, or of the side a (fig. 73).

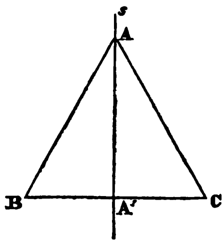
FIG. 73.



A line which joins a vertex to the mid point of the opposite side is called a *median line*. Hence a triangle has three median lines, AA', BB', CC'.

§ 160. **Triangle with Axis of Symmetry.**—Let us now see whether triangles exist which are symmetrical either with regard to an axis or with regard to a centre.

FIG. 74.



If s is an axis of symmetry, A any point on it, and B and C any two points corresponding with regard to the axis, then the triangle ABC will have s for an axis of symmetry. If this axis cuts the side BC at A', then A' is the mid point of BC, or AA' is a median line.

Hence a triangle may have a median line as axis of symmetry.

This is the only case possible; for if a triangle ABC has an axis of symmetry, then to every vertex must correspond a vertex. Hence if B corresponds to C the third vertex must correspond to itself, or it must lie on the axis. But the axis bisects the join of two corresponding points B and C, and is therefore *necessarily* a median line.

If we try in the same manner to construct a triangle having a centre of symmetry, we do not succeed. For to a side there corresponds either another side, which in that case is parallel to the first (§ 153, *b*), or the side corresponds to itself and passes through the centre. But a triangle cannot have two sides parallel, nor can it have all its sides passing through the same point. Hence

A triangle cannot have a centre of symmetry.

§ 161. A triangle which has an axis of symmetry is called a *symmetrical triangle*, or an *isosceles* triangle. We shall use only the former name. That vertex through which the axis passes is called *the vertex*, the opposite side is called the *base*, the remaining sides are called simply the *sides*, and the segment on the axis between the vertex and the base is called the *altitude*, of the triangle.

The principal properties of a symmetrical triangle follow immediately from the construction in § 160. The one vertex A (fig. 74) lies on the axis of symmetry; the others correspond to each other. Hence the lines AB and AC are corresponding lines; therefore they are equal. The same holds for the angles at B and C, for the angles at A, and for those at A'.

These properties may be stated thus:

Theorem: *A symmetrical triangle has*

1st. *Two sides equal, viz. those which meet at the vertex.*

2nd. *Two angles equal, viz. those which are opposite the equal sides.*

3rd. *A median line bisects the angle at the vertex.*

4th. *A median line is perpendicular to the base.*

5th. *The perpendicular bisector of the base passes through the vertex.*

§ 162. Every one of these properties conditions the others, for it can in each case be proved that the triangle has an axis of symmetry. The following are the more important cases :—

Theorem: *A triangle is symmetrical, and has therefore all the properties stated in* § 161—

1st. *If it has two equal sides.* For the bisector of the angle contained by these sides is an axis of symmetry.

2nd. *If it has two equal angles.* For the perpendicular bisector of the side adjacent to these angles is an axis of symmetry.

3rd. *If a median line is perpendicular to the side which it bisects.*

4th. *If the perpendicular bisector of a side passes through the opposite vertex.*

In the last two cases the perpendicular bisector of one side passes through the opposite vertex. It is therefore an axis of symmetry of that vertex, and also of the end points of the side which it bisects, hence of the whole figure.

§ 163. **The Regular Triangle.**—If a triangle has all three sides equal, each of the three median lines is an axis of symmetry, and all angles are equal. Conversely, if a triangle has all angles equal it has all its sides equal.

Such a triangle is called an *equilateral* or *equiangular*, or better a *regular*, *triangle.*

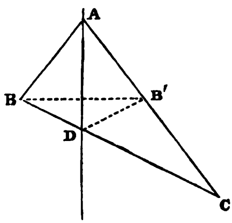
§ 164. Let us now suppose a triangle ABC (fig. 75) in which the bisector of the angle at A is not an axis of symmetry. Then the contra-positive form of the theorem of § 162 tells us that AB is not equal to

$A\,C$, that the angle B is not equal to the angle C, and that the bisector $A\,D$ of the angle at A is not perpendicular to $B\,C$, and hence, that the two angles $A\,D\,B$ and $A\,D\,C$ are unequal. Between these angles there exists the relation

$$\angle B + \angle B\,D\,A = \angle C + \angle C\,D\,A,$$

for each sum makes with half the angle at A an angle of continuation. Hence it follows that, if angle B is greater than angle C, angle $B\,D\,A$ is less than angle $C\,D\,A$.

FIG. 75.



If we now fold the figure along $A\,D$, then $A\,B$ will fall along $A\,C$; and B will fall between A and C if we suppose that $A\,B$ is the shorter of the two unequal lines $A\,B$ and $A\,C$. The line $D\,B$ therefore takes the position $D\,B'$ within the angle $A\,D\,C$. But the angle $A\,B'\,D$, which is equal to angle B, is exterior to the triangle $D\,C\,B'$, and therefore greater than the angle at C (§ 138).

Conversely, if the angle $A\,D\,B < A\,D\,C$, the line $D\,B$ will fall within the angle $A\,D\,C$, and therefore B will fall between A and C, that is, $A\,B$ will be less than $A\,C$. This always happens (see above) if $\angle B > \angle C$, for then $B\,D\,A < A\,D\,C$.

Theorem: *In every triangle the greater side is opposite to the greater angle*, and conversely, *the greater angle is opposite to the greater side.*

In a right-angled triangle, for instance, the hypothenuse is greater than either side.

§ 165. If we join BB' (fig. 75) we get a triangle $BB'C$, in which $\angle DBB' = \angle DB'B$;

therefore $\angle CBB' < CB'B$;

therefore $B'C < BC$.

But $B'C = AC - AB$. Hence

Theorem: *In every triangle the difference of two sides is less than the third side.*

$$AC - AB < BC.$$

From this it follows at once that

$$AB + BC > AC.$$

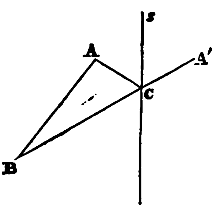
Theorem: *In every triangle the sum of two sides is greater than the third side.*

The proof given for this fundamental theorem required that the two sides AB and AC should be unequal. But if they are equal the theorem does not require a proof, as a little consideration will show.

§ 166. From the last theorem we can draw an important conclusion relating to inequalities in axial symmetry.

Let A and A' (fig. 76) be two points corresponding with regard to the line s as axis of symmetry, and let B be any other point on the same side of the axis as A. Then the line joining B to A' will meet the axis at a point C, so that $AC = CA'$. We have therefore $BC + CA'$ or $BA' = BC + CA$. But $BC + CA > BA$; therefore $BA' > BA$. This may be expressed as follows:

FIG. 76.



Theorem: *Of two points A A' corresponding with regard to an axis of symmetry, that one is the nearer to any point B which lies on the same side of the axis as B.*

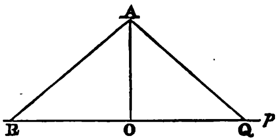
Conversely: *If a point B is at a shorter distance from A than from the point A' which corresponds to A with regard to an axis, then A and B lie on the same side of the axis.*

As a corollary we get

Of two sides in a triangle that is the greater which is cut by the perpendicular bisector of the third side.

§ 167. **Obliques.**—From a point *A* (fig. 77) without a line *p* only one perpendicular can be drawn to that line. If *A O* is the perpendicular, then every other line *A Q* through *A* cuts the line *p* under an acute angle, as the triangle *A O Q* has a right angle at *O*. Such a line *A Q* is called an *oblique*.

FIG. 77.



As the angle at *O* is greater than the angle at *Q*, it follows that $AQ > AO$. Or

Theorem: *Of all segments drawn to a line from a point without it, that on the perpendicular is the shortest. Its length is called the* DISTANCE *of the point A from the line* p. (In geometry by distance the shortest distance is always meant.)

§ 168. If from *A* (fig. 77) two obliques be drawn to a line *p*, say *A Q* and *A R*, which lie on opposite sides of the perpendicular *A O*, and which cut off equal distances, *Q O*, *O R*, on *p*; then $AQ = AR$, as *the triangle A Q R* is symmetrical with regard to *A O*

as axis of symmetry. It follows that from a point A without a line two equal segments can be drawn to the line. If, on the other hand, two obliques AQ and AR are equal, then AQR is a symmetrical triangle; hence the perpendicular from A to the line p must bisect the side QR, and therefore the two equal obliques cut off equal segments QO, OR.

It follows that

Theorem: *From a point A without a line two, and only two, obliques which are equal to a given segment can be drawn to that line, provided that the given segment be greater than the distance of the point from the line.*

Or, in other words

Theorem: *On a line two, and only two, points exist which have a given distance from a fixed point, provided that the given distance is greater than the distance of the point from the line. These two points are symmetrical with regard to the perpendicular from the point to the line.*

On a line there exists one point only which has its distance from a given point A equal to the distance of A from the line: it is the foot of the perpendicular from A to the line.

There is no point on the line which has its distance from a point A less than the distance of the line from that point.

EXERCISES.

(1) If a right-angled triangle is symmetrical, then the axis of symmetry bisects the *right* angle.

(2) An angle in a triangle will be acute, right, or obtuse according as the median line through the vertex of that angle is greater than, equal to, or less than half the opposite side.

(3) A median line of a triangle will be greater than, equal to,

or less than half the side it bisects, according as the angle opposite to that side is acute, right, or obtuse.

Show that this theorem is a logical consequence of its converse, which is contained in the last exercise.

(4) On a given line where is the point whose distances from two fixed points, not on the line, have the least possible sum?

Where is the point whose distances from the two fixed points have the greatest possible difference?

The distances in both cases are to be taken without regard to their sense. How ought the question to be worded if account were taken of the sense of the distances?

N.B. In each of the last two questions two cases at least will have to be considered, inasmuch as the fixed points may be on the same side or on different sides of the line.

(5) The distances of the extremities of the base of an isosceles triangle from the opposite sides are equal to one another.

(6) Through the vertex A of a triangle ABC a straight line XY is drawn perpendicular to the bisector of the angle A. Prove that if M is any point on XY, the perimeter of BMC is greater than that of ABC.

(7) Prove that the three axes of symmetry of a regular triangle meet in a point.

(8) The sum of two sides of a triangle is greater than the sum of the segments joining a point within the triangle to the end points of the third side.

(9) The perimeter of any convex polygon is less than that of any other polygon by which it is completely surrounded.

(10) The sum of the distances of the vertices of a triangle from any point within its area is less than the sum of the sides (or *perimeter*) of the triangle, but greater than half that sum.

(11) The sum of any two sides of a triangle is greater than twice the concurrent median line.

(12) The sum of the three median lines of a triangle is less than the sum of its sides, but greater than half this sum.

(13) Through a point A three half-rays a, b, and q are drawn, a and b being on the same side of q, and making acute angles *with it. Prove* that of the rays a and b that has the greater

distance from any point B in q which makes the greater angle with q.

State and prove also the converse theorem, and investigate the case when a and b are on opposite sides of q.

(14) If two half-rays are symmetrical with regard to a line q as axis, they are equidistant from any point B in q.

To this theorem write down the contra-positive, the converse and its contra-positive, and prove one of the latter two.

Discuss the logical connection between this and exercise (13).

DRAWING.

The following constructions should be made as far as possible by set squares only, without using a pair of compasses.

(15) Construct an isosceles triangle, having given

1. Half the angle at the vertex and the length of either the side, or the altitude, or the base.

2. The angle at the vertex and the length either of the side, or the altitude, or the base.

3. An angle at the base and the length either of the side, or the base, or the altitude.

4. The length of the base and the length of either the side or the altitude.

(16) Construct a regular triangle, having given the length either of a side or the altitude.

(17) Draw the figures to the exercises (1) to (14).

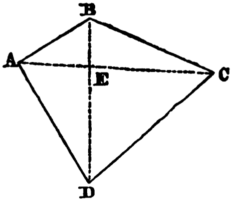
CHAPTER IX.

SYMMETRICAL QUADRILATERALS.

§ 169. By a quadrilateral will be understood at present a polygon of four sides of which no two sides intersect: it may be convex (fig. 78), or it may have a re-entrant angle (fig. 79). Every quadrilateral has

a vertex opposite to every vertex, an angle opposite to every angle, and a side opposite to every side.

FIG. 78.

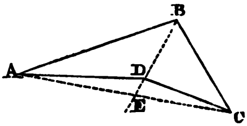


Every side is *adjacent* to two angles and to two sides.

Every angle is *adjacent* to two sides and to two angles.

A quadrilateral has two diagonals; they join opposite vertices. A diagonal may lie either within or without the quadrilateral.

FIG. 79.



If two opposite sides are parallel, the line joining their mid points is called a median line.

Every quadrilateral is divided by each diagonal into two triangles.

In every quadrilateral the sum of the angles is equal to four right angles (§ 140).

§ 170. A quadrilateral may have an axis or a centre of symmetry.

If a quadrilateral has an axis of symmetry, then to every vertex not on the axis there corresponds a vertex not on the axis. Hence the number of vertices not on the axis must be even: either two or all four vertices are off the axis. Hence also the number of vertices on the axis must be even: either two or no vertices lie on the axis. All four vertices cannot lie on the axis, for four points in a line cannot form a quadrilateral.

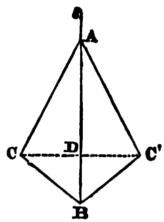
There are therefore two cases of quadrilaterals *which have* an axis of symmetry according as two or

no vertices lie on the axis of symmetry. These quadrilaterals have special names. In the first case the quadrilateral is called a *kite*; in the second case it is a *symmetrical* or *isosceles trapezium*. These have to be considered separately.

§ 171. **The Kite.—Definition**: *A quadrilateral which has a diagonal as axis of symmetry is called a* KITE.

Let A and B be the two vertices on the axis, and CC' the other two. Then C and C' are corresponding points with regard to the axis AB; therefore the line CC' is perpendicular to AB and is bisected by it. This proves the first part of the theorem on page 116.

FIG. 80.



The diagonal point D may lie within or without the kite. In the first case (fig. 80) the kite is convex, and every angle is less than an angle of continuation. In the other case (fig. 81) the figure has a re-entrant angle at B, and the angle CBC' *in* the quadrilateral is greater than an angle of continuation. Intermediate between these we have the case where D coincides with B. The figure is then a symmetrical triangle, which thus appears as the limiting case of a kite.

FIG. 81.

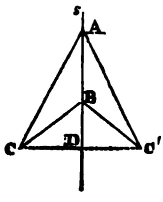
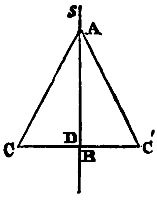


FIG. 82.



In the two other cases the kite is divided by the *transverse axis* into two symmetrical triangles, which

lie on the same or on opposite sides of the transverse axis as common base.

Other properties follow from the symmetry of the figure if we remember that corresponding angles and segments are equal. This gives the following theorem.

Theorem : *A kite has the following properties:*

1st. *One diagonal, the axis, is the perpendicular bisector of the other, which will be called the transverse axis.*

2nd. *The axis bisects the angles at the vertices which it joins.*

3rd. *The angles at the end points of the transverse axis are equal, and equally divided by the latter,* they being corresponding angles.

4th. *Adjacent sides which meet on the axis are equal.*

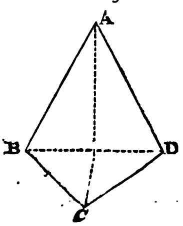
5th. *The axis divides the kite into two triangles which are congruent, with equal sides adjacent.*

6th. *The transverse axis divides the kite into two triangles, each of which is symmetrical.*

7th. *The median lines meet on the axis and are equally inclined to it.*

§ 172. **Conditions that a Quadrilateral may be a Kite.**—*Each of the seven properties enumerated in* § 171 *involves the others, for any quadrilateral which has one of these properties is a kite.*

FIG. 83.



1st. *A quadrilateral is a kite if one diagonal is the perpendicular bisector of the other.*

For it has that diagonal as axis of symmetry.

2nd. *A quadrilateral is a kite if one diagonal bisects the angles at the vertices which it joins.*

For this diagonal is an axis of symmetry, the sides being in pairs equally inclined to it.

3rd. *If each side is equal to one of its adjacent sides.*

Proof: If $AB=AD$ and $CB=CD$ (fig. 83), then the two triangles ABD and CBD are symmetrical, with AC as common axis of symmetry.

4th. *If one diagonal divides it into two isosceles triangles, or if it is made up of two isosceles triangles on a common base.*

For the same reason as in the 3rd case.

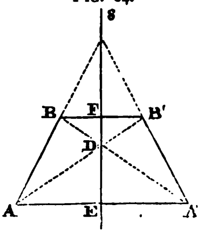
5th. *If two adjacent sides are equal, and if the angle contained by them is bisected by a diagonal.*

Proof: If $BAC=CAD$ (fig. 83), then AB and AD are corresponding lines; and if, further, $AB=AD$, then B and D are corresponding points with regard to AC as axis of symmetry.

The other cases will be left as exercises to the student.

§ 173. **The Symmetrical Trapezium.**—If a quadrilateral has an axis of symmetry which is not a diagonal, then the vertices correspond in pairs—say, A to A' and B to B', A and B being taken as points on the same side of the axis. The lines AA' and BB', as lines joining corresponding points, are perpendicular to the axis of symmetry and are bisected by it.

FIG. 84.



The two lines AA' and BB' are therefore parallel. They are, further, opposite sides of the quadrilateral.

The axis of symmetry thus bisects opposite sides, or it is a median line.

Definition : *A quadrilateral which has a median line as an axis of symmetry is called a* SYMMETRICAL TRAPEZIUM.

Theorem : *The symmetrical trapezium has the following properties :*

1st. *Two opposite sides are parallel, and have a common perpendicular bisector.*

2nd. *The other two opposite sides are equal, and equally inclined to either of the other sides.*

3rd. *Each angle is equal to one, and supplementary to the other, of its two adjacent angles.*

4th. *The diagonals are equal and divide each other equally.*

5th. *The one median line bisects the angle between the two diagonals, and likewise the angle between those two sides produced, which it does not bisect.*

6th. *The other median line bisects the two diagonals, and is parallel to the two sides which it does not bisect.*

7th. *The two median lines are each the perpendicular bisector of the other.*

§ 174. **Conditions that a Quadrilateral be a Symmetrical Trapezium.**—The converse of each of these propositions, with the exception of the last two, holds—that is, any quadrilateral which has one of the first five properties is a symmetrical trapezium.

Theorem: *A quadrilateral is a symmetrical trapezium :*

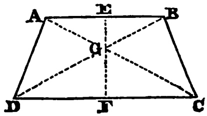
1st. *If two opposite sides have a common perpendicular bisector.*

For this bisector is an axis of symmetry.

2nd. *If two sides are parallel and the angles adjacent to either of them equal.*

Proof: Through the mid point E (fig. 85) of one of the parallel sides AB draw a perpendicular EF to it. This will be an axis of symmetry for the two parallels; also for the points A and B. The lines BC and AD, being equally inclined to AB, will be corresponding lines. Therefore C and D are corresponding points. Hence EF is axis of symmetry of the quadrilateral.

FIG. 85.



3rd. *If two opposite sides are equal and include equal angles with one of the remaining sides.*

Proof: As in the second case.

4th. *If each angle is equal to one and supplementary to the other of its adjacent angles.*

Proof: As in the second case.

5th. *If one median line bisects an angle between the two diagonals.*

Proof: Let the diagonals meet in G (fig. 85); then GAB is a triangle in which the median line GE bisects the angle at the vertex. Hence it is symmetrical. Similarly GDC is symmetrical, having GF as axis. Both triangles, therefore, have EF as axis.

6th. *If the diagonals are equal and divide each other equally, so that the segments of the one are equal to those of the other.*

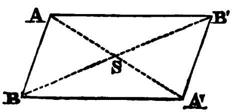
Proof (fig. 85): If $AG=BG$, $GD=GC$, then the bisector of the angles AGB and DGC is an axis of symmetry.

§ 175. **Quadrilateral with Centre of Symmetry.** —*If a* quadrilateral has a *centre* of symmetry, then to .

each side which does not pass through the centre corresponds another side. Hence the number of sides which pass through the centre must be even.

We do not, however, get different figures if we suppose that sides of the quadrilateral either do or do not pass through the centre. For let A, A', B, and B' (fig. 86) be the four vertices, A corresponding to A' and B to B'; then the lines AA' and BB' pass through the centre and are bisected by it.

FIG. 86.



If we now join AB, BA', $A'B'$, and $B'A$, we get a quadrilateral with a centre of symmetry, with those four lines as sides, and with the lines AA' and BB' as diagonals. Here AB and $A'B'$ are corresponding lines; hence they are parallel; and so are likewise BA' and $A'B$.

§ 176. **The Parallelogram.—Definition**: *A quadrilateral which has a centre of symmetry is called a* PARALLELOGRAM.

Theorem: *A parallelogram has the following properties:*

1st. *Opposite sides are parallel.*

2nd. *Opposite sides are equal.*

3rd. *The diagonals bisect each other.*

4th. *Opposite angles are equal.*

5th. *The median lines pass through the centre and are parallel to those sides which they do not bisect.*

§ 177. **Condition that a Quadrilateral may be a Parallelogram.**—The converse to each of the first four propositions is true, that is to say, every quadrilateral which has one of the properties expressed in the first

four propositions is a parallelogram, and has therefore the other properties.

Theorem: *A quadrilateral is a parallelogram.*

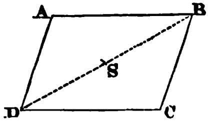
1st. *If opposite sides are parallel.*

Proof: The mid point of any transversal is a centre of symmetry with regard to two parallel lines (§ 123); hence the mid point of a diagonal is a centre of symmetry of the quadrilateral.

2nd. *If two opposite sides are parallel and equal.*

Proof: If AB (fig. 87) is equal and parallel to CD, then the mid point S of the diagonal BD is a centre of symmetry of the parallels AB and CD, B and D being corresponding points. Further, as $BA = CD$, A and C will be corresponding points. Therefore S is a centre of symmetry of all four vertices, and therefore of the quadrilateral.

Fig. 87.



3rd. *If each side is equal to its opposite side.*

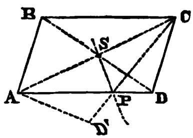
Proof: Let $ABCD$ (fig. 88) be the quadrilateral $AB = CD$, $BC = DA$. Take the perpendicular bisector SP of AC, and turn triangle ACD about it. Let D fall on D'. Then $AD' = AB$, as each equals DC, and $CD' = CB$, as each equals AD. Hence $ABCD'$ is a kite (§ 172, 3).

This shows that

$$\angle ACD = \angle CAB,$$

each being equal to the angle CAD'. The two lines AB and CD are therefore parallel.

Fig. 88.



The quadrilateral $ABCD$ has consequently two opposite sides, AB and CD, equal and parallel. Hence it is a parallelogram according to the 2nd case.

4th. *If the diagonals bisect each other.*

Proof: The intersection of the diagonals is a centre of symmetry for the end points of each; therefore, &c.

5th. *If the quadrilateral is convex and has opposite angles equal.*

Proof: As opposite angles are equal, two adjacent angles are together equal to the remaining two angles, and are hence equal to an angle of continuation. Consequently each side is parallel to the opposite one.

§ 178. We thus see that there exist three distinct symmetrical quadrilaterals, two with an axis and one with a centre of symmetry. The axis is either a diagonal or a median line. The three cases are—

I. *Kite* with a *diagonal* as axis.

II. *Symmetrical trapezium* with a *median line* as axis.

III. *Parallelogram* with *centre.*

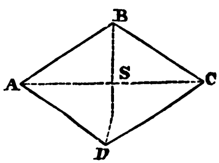
But a quadrilateral may have both an axis and a centre.

This gives two cases.

§ 179. **The Rhombus.**—A kite may have a centre, that is, it may be a parallelogram.

Definition: *A kite with a centre is called a* RHOMBUS.

FIG. 89.



A rhombus (fig. 89) has therefore all the properties of a kite and of a parallelogram. The special properties which result are

1. *Every diagonal is an axis of symmetry.*

For the diagonals of the kite now bisect each other, and hence each is the *perpendicular bisector of the other.*

2. *All its sides are equal.*

Adjacent sides are equal, as each diagonal is an axis.

3. *Each diagonal bisects the angles at the vertices which it joins.*

The converse of each of these propositions holds.

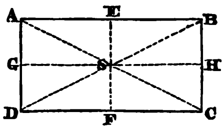
The proof is left to the reader.

§ 180. **The Rectangle.**—A symmetrical trapezium may have a centre or may be a parallelogram.

Definition : *A symmetrical trapezium with a centre is called a rectangle.*

The special properties of the rectangle (fig. 90) are

FIG. 90.



1. *All its angles are right angles.* For opposite angles are supplementary and equal.

2. *The diagonals are equal and bisect each other.*

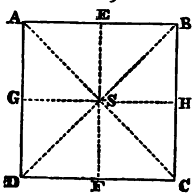
3. *Each median line is an axis of symmetry.*

The converse of each of these propositions holds.

The proof is left to the reader.

§ 181. **The Square.**—Lastly, a quadrilateral may be both a rectangle and a rhombus.

FIG. 91.



Definition : *A quadrilateral which is a kite, a symmetrical trapezium, and a parallelogram is called a* SQUARE.

The special properties of the square follow from those of the rectangle and rhombus.

1st. *All sides are equal and all angles are equal.*

2nd. *Each diagonal and each median line is an axis of symmetry.*

3rd. *The diagonals are equal, and each is the perpendicular bisector of the other.*

4th. *The median lines are equal, and each is the perpendicular bisector of the other.*

The converse of each of these propositions holds.

The proof is left to the reader.

EXERCISES.

(1) The quadrilateral formed by the bisectors of the interior and exterior angles at the base of a symmetrical triangle is a kite.

(2) The quadrilateral formed by the bisectors of the interior angles of a symmetrical trapezium is a kite with two right angles.

(3) In a symmetrical trapezium the mid point of each of the parallel sides is joined to the vertices in the opposite side. Prove that the quadrilateral formed by the joining lines is a kite.

(4) The points in which the bisectors of the angles at the ends of the transverse axis of a kite cut the sides, or the sides produced, are the vertices of a symmetrical trapezium.

(5) The four lines which connect the mid points of the consecutive sides of any quadrilateral form a parallelogram.

What will be the character of this parallelogram if the original quadrilateral is any one of the six symmetrical quadrilaterals treated of in Chapter IX.?

(6) The three lines two of which connect the mid points of opposite sides of a quadrilateral, and the third the mid points of its diagonals, are concurrent; and each is bisected at the point of concurrence.

(7) The quadrilateral formed by joining the mid points of the sides of an isosceles triangle to the mid point of the base is a *rhombus.*

(8) Prove that a parallelogram is

1. A rectangle if its diagonals are equal.

2. A rhombus if one diagonal is either perpendicular to the other or bisects one of the angles of the parallelogram.

3. A square if its two diagonals are equal, and one is either perpendicular to the other or bisects one of the angles of the parallelogram.

(9) All parallelograms which have two of their sides in the sides of a symmetrical triangle, and one vertex in the base of the latter, have equal perimeters.

(10) The sum of the distances of any point on the base of a symmetrical triangle from the sides is constant.

(11) The sum of the distances of a point within a regular triangle from the three sides is constant.

(12) In what direction must a ball on a billiard table be struck in order that it may return to its original position after rebounding from all four sides of the table?

(13) If a figure has two axes of symmetry at right angles, then every *point* is connected with three others which form a *rectangle*, and every *line* is connected with three lines which together form a *rhombus*.

(14) A **regular polygon** of n sides has n axes of symmetry, which all pass through a common point, called the CENTRE of the polygon, and which is equidistant from the vertices and also from the sides.

If n is odd each axis passes through one vertex and through the mid point of one side. Any two vertices and any two sides correspond to one another with regard to one of these axes.

If n is even the axes are of two kinds; one half of them pass each through two vertices, the other half each through the mid points of two sides. Each axis of the one kind is perpendicular either to one of the same, or to one of the other kind, and the centre of the polygon is a centre of symmetry.

With regard to each pair of rectangular axes, each vertex, together with three others, forms a rectangle, and each side, together with three others, forms a rhombus.

How many such rectangles and how many such rhombi exist?

(15) The rays drawn from the centre of a regular polygon to the vertices form a *regular* pencil of n rays, that is, a pencil in which the angles between consecutive rays are all equal.

The rays drawn from the centre at right angles to the sides form a second regular pencil of n rays whose rays bisect the angles of the first.

DRAWING.

(16) Construct the figures mentioned below, having given

1. Of a kite (α) the transverse axis and the segments into which it divides the axis; (β) the transverse axis and the parts into which it divides one of the angles; (γ) the axis and the angles which it bisects.

2. Of a symmetrical trapezium (α) the length of the parallel sides and the distance between them; (β) the length of the parallel sides and the angle between the other sides produced; (γ) two adjacent sides and the included angle; (δ) the angles between two diagonals and the segments into which one divides the other.

3. Of a parallelogram (α) two sides and an angle; (β) two diagonals and the angle between them.

4. Of a rectangle (α) two sides; (β) a diagonal and a side.

5. Of a rhombus (α) one side and an angle; (β) the two diagonals.

6. Of a square (α) a side; (β) a diagonal.

(17) Make the figures to the different cases of the converse propositions in §§ 172, 174, 177, 179, 180, 181.

CHAPTER X.

CONGRUENCE OF TRIANGLES.

§ 182. ACCORDING to § 150 any two congruent figures may be placed in such a position that they are symmetrical with regard to an axis or a centre, This will enable us to find the conditions which are sufficient to ensure the congruence of two figures. Thus two triangles $A\,B\,C$ and $A'\,B'\,C'$ given in the same plane may be placed in a position of symmetry if they are congruent; and conversely, if they can be placed in such a position, then they are necessarily congruent.

FIG. 92.

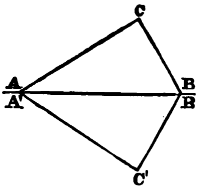
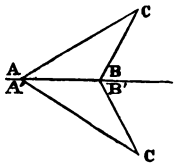
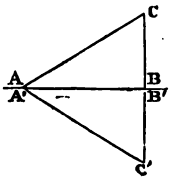


FIG. 93.



§ 183. Let us suppose the two triangles $A\,B\,C$ and $A'\,B'\,C'$ to be congruent, and let us apply the one to the other so that A' coincides with A, B' with B, and C' with C. We may then turn the triangle $A'\,B'\,C'$ over along $A'\,B'$ or $A'B$, whereby we obtain a quadrilateral $A\,C\,B\,C'$, which has the diagonal $A\,B$ as axis of symmetry, and is therefore a kite (figs. 92, 93), or in special cases a symmetrical triangle (fig. 94).

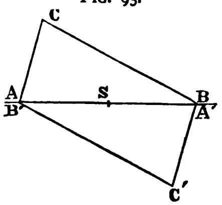
FIG. 94.



Or we may turn $A'\,B'\,C'$ about the mid point S of $A\,B$ through an angle of continuation, till A'

coincides with B and B' with A (fig. 95). The resulting figure is a quadrilateral $ACBC'$, with a centre of symmetry at S. Hence it is a parallelogram.

FIG. 95.



Two congruent triangles can therefore always be placed in either of these two positions. In the first case the two triangles are of opposite sense (§ 132); to make them coincident the one has to be taken *out* of the common plane. In the second case they are of the same sense; they may be made to coincide by merely moving the one triangle *in* their common plane towards the other.

This gives the following theorem:

§ 184. **Theorem**: *Two triangles which are congruent may always be placed in such a position that they together form a kite or an isosceles triangle. If the two triangles are of opposite sense this may be done without taking either triangle out of the plane; but if the two triangles have the same sense, one must be taken out of the plane and turned over.*

Two triangles which are congruent may always be placed in such a position that they form together a parallelogram. If the two triangles are of the same sense this can be done without taking either out of the plane; but if they are of opposite sense one must be taken out of the plane and turned over.

§ 185. That the converse also holds follows from the general investigation of symmetrical figures. The two triangles into which the axis of symmetry divides a kite, or into which a diagonal divides a parallelogram,

are equal, for they are corresponding parts in symmetrical figures.

In order, then, to prove that two triangles are congruent, it is only necessary to show that they may be placed so as to form together a kite, a symmetrical triangle, or a parallelogram.

This requires, first of all, that one side in the one should equal one side in the other. Say $AB = A'B'$, or, using the notation in § 159, $c = c'$. If we then place A on A' and B on B', but make C and C' fall on opposite sides of AB, we have one of the figures 92, 93, or 94.

§ 186. The quadrilateral will be a kite if $AC = AC'$ and $BC = BC'$ (§ 172). Hence

Theorem: *Two triangles are congruent if the three sides of the one are equal respectively to the three sides of the other.*

§ 187. A quadrilateral is also a kite if two adjacent sides are equal, and if the angle between them is bisected by a diagonal (§ 172). This gives

Theorem: *Two triangles are congruent if two sides and the included angle in the one are equal respectively to two sides and the included angle in the other.*

§ 188. A quadrilateral is, thirdly, a kite, if one diagonal bisects the angles at the vertices which it joins. Hence

Theorem: *Two triangles are congruent if one side and the two adjacent angles in the one are equal respectively to one side and the two adjacent angles in the other.*

§ 189. These are the three principal theorems about the congruence of two triangles. In each case

the equality of three elements ensures the equality of the remaining ones.

The cases may be stated thus: In the first case, three sides; in the second, two sides and the included angle; in the third, one side and two angles must be respectively equal.

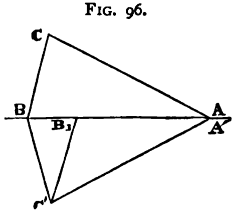
There is one other case possible where three elements are equal, viz. where two sides and an angle opposite one of them, in one triangle, are equal to the corresponding elements in the other. This case requires a different treatment.

Let us suppose we know of two triangles ABC and $A'B'C'$ (figs. 96 and 97) that

$$AC=A'C',$$
$$CB=C'B',$$
$$\text{angle } A=\text{angle } A'.$$

Let us place the two triangles in such a position that A' falls on A, that $A'B'$ falls along AB, and that C and C' fall on opposite sides of AB; then the line AB will bisect the angle CAC', and B' will fall somewhere on the line AB, but on the same side of A as B. Where it falls we do not know; but we do know that AB is an axis of symmetry for the lines AC and AC', that therefore C and C' are corresponding points, as $AC=AC'$. If, therefore, we join C' to B, we get a line $C'B=CB$, since they are corresponding lines.

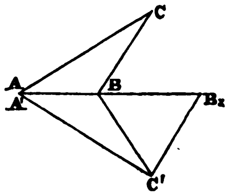
FIG. 96.



But from a point C' without a line AB two lines

may in general be drawn equal in length to a given line CB (§ 168). Of these lines $C'B$ is one. Let $C'B_1$ be the other, then B' must fall either on B or on B_1. In this case, therefore, we cannot assert that the two triangles ABC and $A'B'C'$ are congruent. They may be so, or they may not.

FIG. 97.



In the two triangles $AC'B$ and $AC'B_1$ the angles ABC' and AB_1C' are supplementary, as the triangle $BC'B_1$ is symmetrical. Hence

§ 190. **Theorem** : *If in two triangles two sides and an angle opposite to one of them are equal to the corresponding elements in the other, then the angle opposite the second side in the one is either equal or supplementary to the corresponding angle in the other triangle.*

In the first case the two triangles are congruent.

§ 191. **Corollary I.** : *If the angle opposite the second side is a right angle, the two triangles are always congruent.*

For a right angle is equal to its supplement.

Corollary II. : *If in two triangles two sides and the angle opposite the greater of them are equal to the corresponding elements in the other, the two triangles are congruent.*

For the greater angle is opposite the greater side ; hence if $a' > b'$, $A' > B'$; hence B' must be acute, and its supplement, which is obtuse, cannot be an angle in a triangle satisfying the given conditions.

§ 192. The above are important propositions. The theorem *§ 186,* for instance, tells us that if two triangles

have the sides of the one equal to the sides of the other, then the angles of the one are equal to the angles of the other; and so on for the others.

It must always be borne in mind that those angles are equal which are opposite equal sides.

To these theorems we shall add a few others, which relate to triangles which are not congruent. It will be seen that these follow with equal facility from our investigations of symmetrical figures.

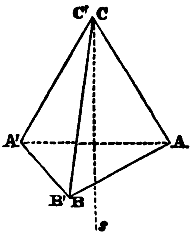
§ 193. If two triangles ABC and $A'B'C'$ have two sides of the one equal to two sides of the other —say, $a=a'$ and $b=b'$, then we know that, if the angles included by these sides are equal, i.e. if $\angle C=\angle C'$, the triangles are congruent (§ 187), hence the third sides are equal, that is,

$$\text{if } C=C' \text{ then } c=c'.$$

We further know that if the third sides are equal then the triangles are congruent (§ 186), and therefore the angles included by the first sides are equal; or

$$\text{if } c=c', \text{ then } C=C'.$$

FIG. 98.



It follows if C is not equal to C', then c cannot be equal to c'. It remains to decide which is the greater.

To investigate this let us suppose that $a=a'$, $b=b'$, and that $\angle C>\angle C'$; and let us place the two triangles in such a position that the equal sides $C'B'$ and CB coincide, whilst the triangles lie on opposite sides of BC (fig. 98). If we now join AA' we have a

symmetrical triangle ACA', CA being equal to CA', with the bisector of the angle at the vertex C as axis of symmetry. But the angle CA' is the sum of two unequal angles; its bisector, therefore, lies within the greater (§ 102), viz. within the angle ACB. Hence the point B lies on the same side of the axis of symmetry as A', and therefore its distance from A' is less than that from A (§ 166), or $A'B < AB$. That is to say,

Theorem: *If two triangles have two sides of the one equal to two sides of the other, but the included angles unequal, then the third sides, opposite these angles, are unequal, and that is the greater which is opposite the greater angle.*

§ 194. The converse also holds—

Theorem: *If two triangles have two sides of the one equal to two sides of the other, and the third sides unequal, then the angles opposite these third sides are unequal, and that is the greater which is opposite the greater side.*

The proof is the same as in the former theorem. But it may also be considered as a logical consequence of the last together with the theorem in § 186; for the angles in question cannot be equal, as this involves the congruence of the triangles. Neither can the greater angle be opposite the smaller side. This would contradict the theorem in § 193.

Exercises.

(1) Two triangles are congruent (α) if two sides and a median line in the one are equal respectively to two sides and a corresponding median line in the other (two cases), (β) if the three median lines in the one equal those in the other.

(2) In each of the following cases prove that the two figures named are congruent, if they have the elements mentioned respectively equal:

1. Two kites: (α) one side and the two adjacent angles; (β) two adjacent unequal sides and the included angle.

2. Two symmetrical trapeziums: (α) the four sides; (β) two adjacent sides and the included angle; (γ) two adjacent sides and a diagonal; (δ) the two parallel sides and their distance.

3. Two parallelograms: (α) two adjacent sides and a diagonal; (β) two diagonals and the included angle; (γ) two sides and the included angle.

4. Two rhombi: (α) one side and one angle; (β) two diagonals; (γ) one side and one diagonal.

5. Two rectangles: (α) two adjacent sides; (β) a side and a diagonal; (γ) two diagonals and the included angle.

6. Two squares: (α) one side; (β) one diagonal.

(3) Two quadrilaterals $ABCD$ and $A'B'C'D'$ are congruent if $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, $DA = D'A'$, and if the angle at A equals the angle at A', provided that both quadrilaterals are convex, that both have a re-entrant angle, or that in each one side meets another.

(4) Two regular polygons of the same number of sides are congruent if a side in the one equals a side in the other.

(5) If through a row of equidistant points in a line parallel lines be drawn, then every line will be cut by these parallels in a row of equidistant points, and the parallels will be themselves equidistant.

DRAWING.

(6) Divide a given segment into n equal parts. [See Ex. (5).]

(7) Construct a right-angled triangle, having given (α) a side and the opposite angle, (β) a side and the segment adjacent to it on the hypothenuse terminated by the perpendicular from the vertex of the right angle, (γ) the hypothenuse and the sum of the two sides.

(8) Construct a regular triangle having a perimeter of given length.

(9) Construct, by aid of set squares and of a pair of compasses, regular convex polygons of three, four, six, eight, twelve sides, having a side of given length.

(10) Construct out of the regular convex polygons all possible regular star polygons of six, eight, or twelve sides (α) by diagonals, joining each vertex to the next but one, or to the next but two; (β) by the sides, producing each side till it meets the next but one or the next but two.

CHAPTER XI.

LOCI OF POINTS AND SETS OF LINES.

§ 195. THERE are in a plane an infinite number of points, and also an infinite number of lines. If it is only required of a point that it shall lie in a plane, we have the choice between all points in the plane. But if it is required of the point that it shall lie in a line, then our choice, though still lying between an infinite number of points, is restricted to a much narrower field. It has been reduced from a spread of points of two dimensions to one of one dimension only. If we likewise know that the point lies on another given line, it is perfectly determined, for the two lines have but one point in common.

Reciprocally, a line in a plane leaves us a choice between the elements of a two-way spread. But if the line in a plane is required to pass through a given point A in the plane, it must be a line in the pencil which has A as centre. Our choice is therefore limited to a one-way spread of lines.

If we further know that the line passes through

another point B, it must lie in a second pencil with B as centre, and is now perfectly determined, for two points have but one line in common.

Hence the condition that a point shall lie on a line, and, in the reciprocal figure, the condition that a line shall pass through a point, is of such a character that *two* conditions of this kind fix the position of the point or of the line in a plane.

§ 196. Instead of saying that a point shall lie in a line, we may say that the point shall be one of the points in a row, and instead of saying that the lines shall pass through a point, we may say that it is a ray in a pencil. Two rows always have one element in common unless their bases are parallel. In the latter case they have in general no point in common ; but they have all in common if they have one in common, for then the bases coincide.

Two pencils always have one ray in common if both pencils are central pencils, or if the one is a central and the other a pencil of parallels. If both are pencils of parallels, they have in general no common ray ; but they have all rays common if they have one in common, for the two pencils will in this case have the same direction, and will therefore coincide.

§ 197. Similarly if it is required that a point shall lie in a given line and have a given distance from a given point, we know (§ 168) there are two points which satisfy both conditions. Here again *two* conditions fix the position of a point.

As we proceed we shall see that there are *always two conditions* required to determine the position of

a point or of a line, or at least to leave us only the choice between a finite number of points or lines.

§ 198. To resume our investigation, if one condition only is given, there will be an infinite number of points or lines which satisfy this condition.

All these points or lines will, as a rule, follow continuously, and will constitute a one-way spread of points or lines.

This spread of points or lines forms a figure which has the property that every point or line belonging to it satisfies a given condition, and conversely that all points or lines satisfying the given condition belong to it.

In other words, it contains *all* the points, or lines, which satisfy a given condition, and *only* such points or lines.

Instead of a *condition* we have a *figure* which replaces it; and this figure is called in the one case a *locus of points*, in the other a *set of lines, satisfying the given condition.*

§ 199. The investigations of loci of points and of sets of lines are reciprocal, and may therefore be conveniently treated together. This will be done, as in the case of axial and central symmetry, by printing corresponding propositions in columns opposite to each other on the page. Thus we get

Definition: *The aggregate of all points in a plane satisfying a given condition is called the* LOCUS *of points satisfying that condition.*

Definition: *The aggregate of all lines in a plane satisfying a given condition is called the* SET OF LINES *satisfying that condition.*

§ 200. **Example of Loci and Sets.**—An example will illustrate this.

Problem: Let it be required to find a point P which shall have the property that the line joining it to a fixed point A shall pass through another fixed point B, then the point P must lie in the line joining AB; and as every point on this line satisfies the condition, it follows that the required locus is the line joining A to B.

If the given points A and B are two distinct points, this locus always exists.

Problem: Let it be required to find a line p which shall have the property that the point joining it to a fixed line a shall lie in another fixed line b, then the line p must lie in the join ab; and as every line in this point satisfies the condition, it follows that the required set is the pencil, with centre on the join of a and b.

If the two given lines a and b are two distinct lines which are not parallel, this set always exists.

§ 201. If, in the second problem, two lines a and b are parallel, then the set consists of a pencil of parallels—viz. of lines parallel to a and b. For the line p is required to cut a where b cuts a. But b does not cut a if they are parallel; therefore p must not cut a.

We see thus that the pencil of parallels appears here as a special case of the pencil of lines passing through a point, and that therefore the name *pencil* of parallels is justified.

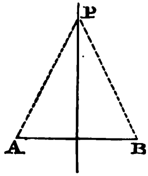
§ 202. **Problem** a: *To find the locus of points*

Problem b: *To find the set of lines which are*

vhich are equidistant from wo given points. *equally inclined to two given lines.*

To solve the first problem let A and B be the ;iven points (fig. 99), and let P be any point satisfying he condition. Then $PA=PB$. It ollows at once that APB is a symnetrical triangle, which has AB as ›ase, and P as vertex, and further that he vertices of all such triangles satisfy he condition.

FIG. 99.



But we know (§ 161) that the ›erpendicular from the vertex of a ymmetrical triangle to the base is the perpendicular ›isector of the base; and as there is only one such ›isector, it follows that *all* points satisfying the given :ondition lie on the perpendicular bisector of the ›ase. We know also that *every* point on the per›endicular bisector of the line AB is equidistant rom the two points A and B, for A and B are ymmetrical with regard to the perpendicular bisector ›f the segment AB. Hence

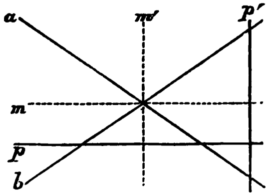
Theorem: *The perpendicular bisector of the seg'ent AB is the locus of points equidistant from the wo points A and B.* For this line contains *all* points, nd *only* points, which satisfy the given condition.

It is therefore the same thing whether we say that point P shall be equidistant from two given points A and B, or that the point P shall lie in the perpen.icular bisector of AB. Thus we have replaced the 'iven condition by a line.

§ 203. To solve the second problem, let a and b fig. 100) be the given lines, which we will suppose

not to be parallel, and let p be any line satisfying the condition. Then $\angle pa = \angle pb$. It follows at once, since apb has two equal angles, that it is a symmetrical triangle, having ab as vertex and p as base, and that all bases of such triangles satisfy the condition. If we draw through the vertex of this triangle a line parallel to the base p, this will bisect the angle ba, because this line also will be equally inclined to a and b. It follows that every line which satisfies our condition is parallel to a bisector of an angle between a and b.

FIG. 100.



But also, conversely, every line which is parallel to such a bisector makes equal angles with the two lines, and therefore belongs to our set. Now, there are two such bisectors, and these are at right angles the one to the other; hence

Theorem: *The set of lines equally inclined to two given lines which are not parallel consists of two pencils of parallels. Each pencil is parallel to one of the bisectors of the angles between the two lines. The directions of the two pencils are therefore perpendicular to one another.*

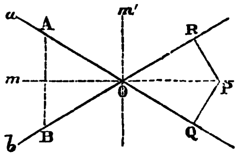
It is therefore the same thing whether we say that a line p shall be equally inclined to two given lines a and b, or that the line p shall belong to a pencil of parallels, parallel to one of the bisectors between the angles.

§ 204. **Problem a**: *To find the locus of points which have equal distances from two given lines.*

Problem b: *To find the set of lines which have equal distances from two given points.*

To solve the first of these problems, the given figure consists of two lines, a and b, which in general will intersect at a point—say, O. If P (fig. 101) be a point satisfying the condition, then the perpendiculars PQ and PR from P to the two lines are equal. But then the right-angled triangles OPQ and OPR are congruent (§ 191), because they have the hypothenuse in common, and the two sides PQ and PR equal. Hence the angle ROP will be equal to POQ; so the line OP bisects the angle at O, and is therefore known. This bisector is an axis of symmetry, and therefore every point in it is equidistant from a and b. Hence every point satisfying the given condition lies on a bisector of an angle between a and b, and every point in such a bisector satisfies the condition.

FIG. 101.



Now, there are two lines, and only two lines, which bisect the angles between a and b. It follows that—

Theorem: *The locus of points equidistant from two given lines consists of the two lines which bisect the angles between the given lines.*

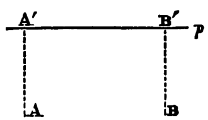
These two lines are perpendicular to one another (§ 110). The locus consists here of two lines, which *together* constitute it. Two lines which go together in such a manner are generally called a *line-pair*.

If the two given lines are parallel the locus will consist of *one line only, which lies half-way between the given lines, and is parallel to them.* It is, again, an axis of symmetry for the given lines. The proof of this assertion is left to the student.

It is of interest to ask, What has become of the second line in this case? The student will easily see, if he considers the line m fixed, and supposes a and b to turn about two points A and B symmetrical with regard to m, so that the point O moves to a greater and greater distance, how the line m' also moves to a greater and greater distance, till at last, if a and b become parallel, the point O and the line m' disappear at an indefinite distance out of our reach.

§ 205. To solve the reciprocal problem, let us suppose that p (figs. 102 and 103) is a line satisfying the condition; then the perpendiculars AA' and BB', drawn from the given points to p, will be equal. But they are also parallel, as both are perpendicular to p. Hence the quadrilateral formed by the four points A, B, A', and B' is a parallelogram (§ 177), as the opposite sides AA' and BB' are equal and parallel. Of this parallelogram the line AB will either be a side or a diagonal.

FIG. 102.

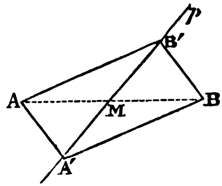


To distinguish these cases we need only observe that the two points must either lie on the same side of p or on opposite sides. In the former case (fig. 102) AB is one side and p is opposite to it, hence parallel to AB.

In the other case (fig. 103) AB will be one

diagonal and p the other; hence p will pass through the mid point M of AB.

FIG. 103.



It follows that any line satisfying the given condition is either parallel to the line AB or passes through the mid point of the segment AB.

We conclude that *all* lines of our set are included in a pencil of lines parallel to AB and in a pencil of lines passing through M, and it only remains to show that *every* line in these pencils satisfies the condition, or that these pencils contain only lines of our set. But this is easily seen.

First, if p be any line parallel to AB, then the perpendiculars from A and B to p are parallel, and constitute, together with AB and p, a parallelogram in which the perpendiculars are opposite sides, and therefore equal.

Secondly, if p be any line through M (fig. 103), then the perpendiculars AA' and BB' to it form with AB' and $A'B$ a quadrilateral, of which M is a centre of symmetry, because AA' and BB' are parallel. Hence it is a parallelogram, and has therefore opposite sides equal, so that $AA'=BB'$. The line p therefore satisfies the condition.

This shows that our set consists of two pencils, the one central, with its centre at the mid point of AB, the other a pencil of parallels to the line AB.

As the solutions to our problems (§ 204) we may, then, state the following two theorems:

§ 206. **Theorem** a: *The locus of points equidistant*

Theorem b: *The set of lines which have equal*

from two given lines which are not parallel consists of the two rows whose bases bisect the angles between the given lines.

If the two lines are parallel, the locus consists of one row whose base bisects the strip between the two parallels.

distances from two given points A and B consists of a pencil of lines having the mid point of A B as centre, together with a pencil of lines parallel to A B.

§ 207. **Method of Intersection of Loci.**—The investigation of loci and sets is of great use in all problems where it is required to find points or lines which satisfy two conditions. For if we leave out one condition, we may find a locus of points, or set of lines, satisfying the other condition.

Thus each condition may be replaced by the corresponding locus or set. If these two loci, or sets, have an element in common, we get points or lines belonging to both loci or sets, and therefore satisfying both conditions. And these will be all the points or lines satisfying both; for if a point does not lie on *both* loci, or if a line does not belong to *both* sets, it will not satisfy *both* conditions.

§ 208. **Problem** a: *Let it be required to find points which shall have equal distances from two given points A and B, and also be at equal distances from two given lines* a, b.

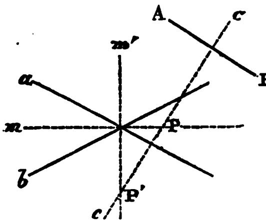
The first condition

Problem b: *Let it be required to find lines which shall be equally inclined to two given lines* a *and* b, *and also be at equal distances from two given points A and B.*

The first condition

demands that the points required lie on the perpendicular bisector of AB. This line we may call c (fig. 104). The second

FIG. 104.

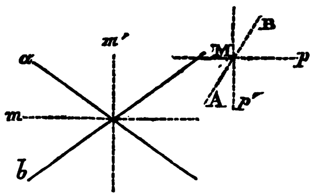


condition gives as locus the line-pair bisecting the angles between the given lines a, b. These two lines may be called m and m'. The line c in general cuts each of the lines m and m' in one point, and in one point only. If P and P' are these points, then they will satisfy both the given conditions. Our problem, therefore, has two solutions; or

There are in general two, and only two, points which are equidistant from two given points and also

demands that the lines required be parallel to one of the bisectors of the angles between the lines a and b. The second

FIG. 105.



condition requires that it be either parallel to the line AB or that it pass through the mid point M of the segment AB. The pencil M has one ray in common with each of the two pencils of parallels, given by the first condition. The pencil of parallels to AB has in general no line in common with the above pencils of parallels.

Hence

There are in general two, and only two, lines which are equally inclined

equidistant from two given lines.

to two given lines, and are equidistant from two given points.

The exceptions which are possible for special positions of the points and lines are left to the student to investigate.

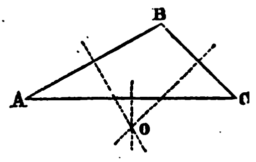
§ 209. More important results are obtained if we ask

Problem a: *To find points equidistant from three given points A, B, and C, which are not in a line, and which therefore form a triangle.*

Problem b: *To find lines equally inclined to three given lines* a, b, *and* c, *which do not pass through a point, and of which no two are parallel, and which therefore form a triangle.*

To answer the first question we take first the locus of points equidistant from A and B (fig. 106), that is the perpendicular bisector of AB, and next the locus of points equidistant from A and C, viz. the perpendicular bisector of AC. These two lines, provided they are not parallel, meet in one point O, and in one point only. They are parallel only if the three points A, B, C lie in a line. Hence

FIG. 106.



§ 210. **Theorem**: *There is one, and only one, point equidistant from three given points which do not lie in a line.*

This point O is equidistant from A and B, and also from A and C, therefore from B and C. But all points equidistant from B and C lie on the perpendicular bisector of BC. This gives the following

Theorem : *The three perpendicular bisectors of the sides of any triangle meet in a point.*

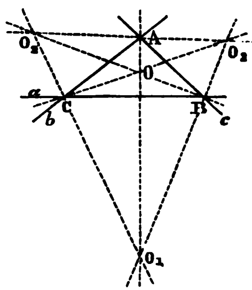
§ 211. The reciprocal problem stated above has no solution, and gives a new example that the principle of duality has exceptions. In the higher parts of geometry these exceptions are gradually removed.

§ 212. **Problem** a: *To find points equidistant from three given lines which form a triangle.*

Problem b: *To find lines equidistant from three given points which form a triangle.*

To solve the first problem, let a, b, c (fig. 107) be the given lines intersecting in the three distinct points A, B, C. The locus of points equidistant from the two lines b and c consists of the two bisectors of the angles between the lines. The locus of points equidistant from the two lines c and a similarly consists of two lines, bisecting the angles between c and a. Our two loci consist thus of two pairs of lines. Each line of the one pair cuts each line of the second pair in one point, so that we get four points O, O_1, O_2, O_3 common to the two loci. Hence

FIG. 107.



Theorem : *There are in general four points which are equidistant from the three sides of a triangle.*

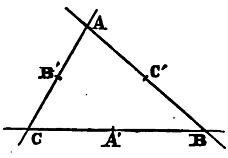
§ 213. Every one of these four points is equi-

distant from b and c, and also from c and a, therefore also from a and b. But all points which are equidistant from the lines a and b lie on the bisectors of the angles between these lines. It follows that the four points O, O_1, O_2, O_3 lie on the bisectors of the angle between a and b, and it is easily seen that each of these bisectors contains two of them. Hence

Theorem: *The bisectors of the interior and exterior angles of a triangle meet four times by threes in a point.*

§ 214. To solve the reciprocal theorem let A, B, C be the three points forming a triangle (fig. 108), and let A', B', C' be the mid points of the sides, viz. A' of BC, B' of CA, and C' of AB. Then the set of lines equidistant from B and C consists of two pencils of lines, the one parallel to BC, the other passing through the mid point A' of BC. Similarly the set of lines equidistant from C and A consists of two pencils, the one parallel to CA, the other passing through the mid point B' of CA. A line belonging to both sets satisfies all conditions. Of these lines there are three. First, the line $A'B'$ belongs to the two pencils A' and B'; hence it gives a solution. The pencil of parallels to BC contains one line through B' and gives a solution. Lastly, the parallels to CA contain one line passing through A'. The two pencils of parallels contain no common ray. Hence

FIG. 108.



Theorem: *There are three, and only three, lines*

which are equidistant from three given points not lying in a line.

§ 215. Each of these three lines is equidistant from A and B, and is therefore either parallel to AB or it passes through the mid point C' of AB.

Through each of the mid points A' and B' of the other sides pass two lines, whilst one line is parallel to each of these sides. Hence we conclude that two of our three lines pass through C', whilst one is parallel to AB. The lines parallel to the other sides cannot be parallel to AB. Hence the line passing through the mid points A' and B' of two sides is parallel to the third ; or

Theorem : *In every triangle the line joining the mid points of two sides is parallel to the third side.*

The three lines equidistant from three points A, B, C are therefore the three sides of a triangle whose vertices are the mid points of the sides of the given triangle.

§ 216. We shall finish this chapter with a determination of the number of lines which pass through a given point P and which are equidistant from another given point S.

If through the point P a line a be drawn, and through the point S a perpendicular to this line, meeting it at Q, then the points Q, S, and P will form a triangle, provided that Q does not coincide with either P or S.

If Q coincides with S, the line a through P must pass through S. In this case the perpendicular from S to the line has no length. If Q coincides with P, the line a must be perpendicular to SP. In this case

the distance of S from a is equal to SP. In every other case we get a triangle SQP with a right angle at Q, so that the side SQ is less than SP, the hypothenuse. The distance of the line a from the point S is therefore less than SP.

In this case the line a', symmetrical to a with regard to the line SP as axis, has the same distance from S as a. And this is the only line which has this property; for if we suppose that a'' is a line having the same distance from S as a, then S must lie on the bisector of an angle between the lines a and a'' (§ 206), that is, the line SP bisects the angle at P, and is therefore an axis of symmetry for a and a''. But there is only one line symmetrical to a with regard to SP, so that a'' must coincide with a'. From all this it follows that

Theorem: *Through a given point P* ONE *line can be drawn which has* NO *distance from another given point S;* TWO *lines whose distances from S are both equal to a segment which is less than the distance SP between the given points;* ONE *line which has its distance from S equal to SP; and* NO *line which has its distance greater than SP.*

EXERCISES.

(1) Find the locus of the mid points of segments drawn from a given point A to a given line which does not pass through A.

(2) Given the sum (or the difference) of the distances of a point from two intersecting lines. Find its *locus*.

(3) Segments are drawn between the sides and parallel to the base of a symmetrical triangle. Find the locus of their mid points.

(4) The points where any two lines parallel to the base of a

symmetrical triangle cut the sides are joined crosswise. Find the locus of the intersection of these joins.

(5) Find the locus of points at a given distance from a given line, the distance having a definite sense.

(6) Two congruent pencils have such a position that the two coincident rays, one belonging to each pencil and passing through the centre of the other, correspond to each other. Find the locus of joins of corresponding rays.

(7) Two congruent rows are placed in such a position that (α) a point in the one coincides with its corresponding point in the other, or (β) that their bases are parallel. Determine the set of lines joining corresponding points. Distinguish two cases (β) according as the rows have the same or opposite sense.

(8) In a given line find a point (α) equidistant from two given points, (β) equidistant from two given lines, (γ) so that the lines joining it to two given points are equally inclined to it. How many such points exist in each case?

(9) In a given point find a line which is (α) equally inclined to two given lines, or (β) equidistant from two given points.

(10) Find a line which shall have equal distances from two given points A_1 and A_2, and also equal distances from two other points B_1 and B_2. How many such lines are possible (α) when the lines A_1A_2 and B_1B_2 are not parallel, or (β) when they are parallel?

(11) State and solve the problem reciprocal to the last.

DRAWING.

(12) Draw the figure to each of the above exercises.

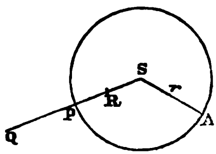
CHAPTER XII.

THE CIRCLE AS LOCUS.

§ 217. **Definition of the Circle.**—*If a segment of a line turns about one of its end points, the other*

end point describes a curve which is called a CIRCLE. *The fixed end point is called the* CENTRE *of the circle, and the moving segment in any position is called a* RADIUS *of the circle.*

FIG. 109.



§ 218. As the moving line, and with it the moving end point, after making an entire revolution, returns to its original position, it follows that the *circle is a closed curve.* It divides the plane into two parts. The one is finite, and is swept over by the moving segment whose end point describes the circle. Any point in this part is said to lie *within* the circle. Any point lying in the remaining part of the unlimited plane is said to lie *without* the circle.

§ 219. Any line drawn from the centre to a point on the circle is a radius, for it is one of the positions of the describing segment. As this segment is of invariable length, we see that *all radii are equal.* Hence *all points on the circle have the same distance from the centre.* This distance is equal to the length of the radius, and is called the *radius distance.*

All points at the radius distance from the centre are on the circle; for the line joining such a point to the centre is a radius of the circle, and the moving segment in describing the circle coincides with it once.

The circle is therefore the locus of points equidistant from a fixed point, the centre.

Any point R within the circle lies on some radius SP, and therefore $SR < SP$.

Similarly if Q is a point without the circle, then SQ contains a radius SP, so that $SQ > SP$; or,

Theorem: *A point lies within, on, or without a circle, according as its distance from the centre is less than, equal to, or greater than the radius distance of the circle.*

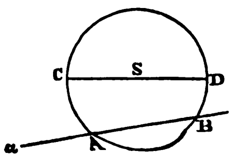
§ 220. **Fundamental Properties of the Circle.**—*Two circles with equal radii are equal, and can be made to coincide if the centre of the one be placed on the centre of the other. They are called equal circles.* For every point on the second circle is in this position at the radius distance from the centre of the first, and therefore on the first.

The second circle may, in this position, be turned about its centre, and still it will coincide with the first. Hence also *a circle can be made to slide along itself by being turned about its centre.* For of the above coincident circles we may consider the one simply as a trace of the other. This property of the circle, that it is a curve which can slide along itself, is its fundamental property. It allows us to turn any figure, connected with the circle, about the centre without changing its relation to the circle. We shall often make use of it.

§ 221. **Theorem**: *A line cannot have more than two points in common with a circle.*

For there are (§ 168) never more than two points in a line which have their distances from any fixed point, here the centre, equal to a given length, here the *radius* distance.

Fig. 110.



§ 222. **Definitions**: A line a which cuts a circle at two points A, B (fig. 110) is called a *secant* of the circle. The segment AB on this line, which is bounded by the circle, is called a *chord* of the circle.

A chord through the centre is called *a diameter* (CD in fig. 110).

§ 223. **Properties of Diameters.**—*Every diameter is bisected by the centre of the circle. Every diameter is equal to two radii.*

All diameters are equal.

A circle cannot have more than one centre.

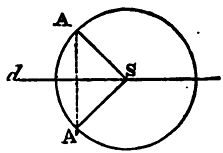
For if it had two, the line joining them would be a diameter having two mid points.

§ 224. **Centre of Symmetry.**—*A centre of the circle is a centre of symmetry, the end points of any diameter being corresponding points.*

This follows at once from the fact that the circle slides along itself when turned about its centre.

§ 225. **Axes of Symmetry.**—*Every diameter is an axis of symmetry.* For if we fold over along a diameter d, every point on the part of the circle turned over must fall on some point on the other, as it is at the radius disance from the centre which remains fixed.

Fig. 111.



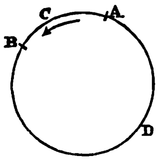
Conversely, *Every line which is an axis of symmetry of a circle is a diameter of the circle.* For, if not, there would be a point symmetrical to the centre, and this too would again be a centre. The circle would thus have two centres, which is contrary to § 223.

The circle has therefore an infinite number of axes

of symmetry, and has besides a centre of symmetry. This, together with the property stated in § 220, that the circle coincides with itself after turning about the centre through *any angle*, allows us to state at once a great number of its properties.

§ 226. **Definitions**: Any two points A, B (fig. 112) on the circle divide it into two parts, which are called *arcs*. If a given point, in describing a circle, moves from one position A to another B, it describes an *arc* AB. If the point moves till it comes back through the same point, it has described the whole circle.

FIG. 112.



This, as an arc, is called the *circumference.*

Arcs are said to be *equal* if they can be made to coincide. Equal arcs may lie either on the same circle or on equal circles.

§ 227. As the arc, like the segment of a line, is described by the motion of a point, it has a *sense.*

In future, unless otherwise stated, the sense will be taken as positive if the describing point appears when seen from the centre, or from any other point within the circle, to move from right to left, as indicated by the arrow in fig. 112.

This determines the arc AB. Otherwise we should not know whether the arc ACB or ADB was meant by AB.

§ 228. **Sum of Arcs of a Circle.**—Between arcs of a circle there exist relations analogous to those between segments on a line. Thus we have

$$AC + CB = AB.$$

$$AB + BA = \text{circumference}.$$

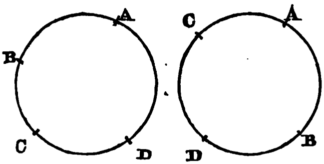
But if we consider only the amount of motion required to come from A to A, we may say this is zero. Just as, in the case of an angle, we could add an angle of rotation any number of times to any given angle without altering the figure, so we may now add to any arc a circumference any number of times, without changing the end points of the arc. But we shall understand by the arc AB, unless otherwise stated, the *smallest arc* described by a point moving in the positive sense from A to B. The arc AA gives, then, either the circumference or the arc zero of no length. Hence we may write

$$AB+BA=o,$$

and $$AB=-BA,$$

where now $-BA$ means the arc described by a point moved along the circle from B to A in the negative sense.

FIG. 113.



The same equations which hold for segments (§§ 75–82) and for angles (§§ 90–95) hold also for arcs of circles. For instance

$$AB+BC+CA=o,$$
$$AB+BC+CD=AD.$$

§ 229. **Angle at the Centre.**—If a point describes the arc AB, the radius drawn to it will describe an angle having its vertex at the centre. This angle is called an *angle at the centre*, and is said to be subtended *by the* arc AB, or to stand upon the arc AB.

§ 230. **Theorem**: *Equal arcs subtend equal angles*

at the centre; and conversely, equal angles at the centre are subtended by equal arcs.

For if we have the arcs AB and CD equal, we may slide the arc CD, together with the radii SC and SD, along the circle till C coincides with A; then D will coincide with B, as $CD=AB$. Therefore the angle CSD will coincide with ASB and will be equal to it in magnitude and sense.

It follows that if $A, B, C\ldots$ denote points on the circle, and $a, b, c\ldots$ the radii drawn to these points, then every equation between arcs AB, BC, &c., will carry with it an equation between the corresponding angles ab, bc, &c., and *vice versâ*.

If $AB=CD$, then $ab=cd$;

and also if $ab=cd$, then $AB=CD$.

In the same manner the equation between arcs

$$AB+BC=AC$$

involves also the equation between angles

$$ab+bc=ac,$$

and so on.

§ 231. A diameter, being an axis of symmetry, divides the circle into two equal arcs, called *semicircles*.

A semicircle subtends an angle of continuation; or, *every diameter as an angle of continuation is subtended by a semicircle.*

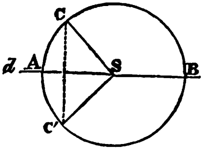
Similarly every right angle at the centre is subtended by half a semicircle, called a *quadrant* of the circle.

§ 232. From the fact that a diameter is an axis of symmetry, other properties follow.

To every point C on the circle corresponds, with

regard to a diameter d as axis of symmetry, another point C' on the circle. The line CC' joining these is therefore perpendicular to d and bisected by it.

FIG. 114.



If we want to find for any point C its corresponding point C' with regard to d as axis, we have to draw from C a perpendicular to d, and to produce it till it cuts the circle again at C'. Then C' is the required point, for the point corresponding to a point on the circle lies itself on the circle. This implies the following

Theorem: *A diameter bisects all chords perpendicular to it.*

Converse Theorem: *If a diameter or a radius bisects a chord which is not itself a diameter, it is perpendicular thereto.* For there is but one diameter that bisects a given chord.

Corollary: *The locus of mid points of parallel chords is a diameter perpendicular to those chords.*

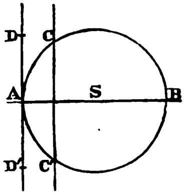
§ 233. If, as in § 232, C, C' are points corresponding with regard to a diameter d (fig. 114), then the arcs CA and AC' will be corresponding arcs, and hence equal. The arc CC', and for the same reason the arc $C'C$, is therefore bisected by the diameter d.

As CA and AC' are equal arcs, the angles at the centre subtended by them are equal; hence angle CSC' is also bisected by d. This proves that

Theorem: *The diameter perpendicular to a chord bisects that chord, bisects the two arcs into which this chord divides the circle, and bisects the angles at the centre subtended by these arcs.*

§ 234. We have seen that if a diameter AB is taken as axis of symmetry, then to every point C on the circle corresponds another point C', such that CC' is perpendicular to AB. This supposes that the point C does not lie on the axis of symmetry or on the diameter AB. Hence if we draw through one of the end points A of the diameter, a perpendicular to it, this will correspond to itself, and thus can have only the one point A in common with the circle. For if it met the circle again at a point D, the point D', symmetrical to D with regard to the diameter AB, would also lie on the circle, or the perpendicular would have three points on the circle, which is impossible (§ 221).

Fig. 115.



Definition of a Tangent.—*A line which has only one point in common with a circle is called a* TANGENT *to the circle, and that point is called the* POINT OF CONTACT.

The above reasoning gives now the following theorem.

Theorem: *A line through the end point of any diameter and perpendicular to it is a tangent to the circle, and has that end point as its point of contact.*

§ 235. A tangent may also be considered from another point of view. If we suppose the chord CC', joining symmetrical points, to move away from the centre towards A, then the two end points C and C' will approach A from opposite sides, and will at last coincide at A. The line CC', which remains always perpendicular to the diameter AB, will now be a *tangent at A.* From this we see that a tangent to a

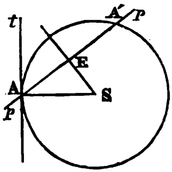
circle may be considered as a line which cuts a circle in *two coincident* points. This also may be taken as a definition of a tangent.

Second Definition of a Tangent.—*If a secant which cuts a circle in two points be moved in any manner till the two points of intersection with the circle coincide, it is in this last position called a tangent to the circle.*

§ 236. At every point A on the circle we can draw a tangent, for we need only draw the radius to A and erect a perpendicular at its end point A. This will be a tangent at A.

But only one tangent at a point A on a circle can be drawn. For if we draw through the point A any other line p not perpendicular to the radius AS, then the perpendicular from the centre S to this line will cut it at a point E different from A. Hence the line p must cut the circle again at some other point A' which is symmetrical to A with regard to the diameter SE as axis of symmetry. Hence

FIG. 116.



Theorem : *At every point on the circle one, and only one, tangent can be drawn to the circle.*

§ 237. This shows also that

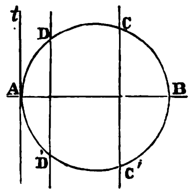
Theorem : *The perpendicular from the centre to a tangent of a circle passes through the point of contact.*

The radius to the point of contact of a tangent is perpendicular to the tangent.

Both propositions follow from the fact that but one tangent can be drawn having a given point on the circle as point of contact; and that there exists but one perpendicular from the centre to a line.

§ 238. To find other properties let us again consider a diameter AB as axis of symmetry, and let CC' and DD' be two pairs of corresponding points, then CC' and DD' are parallel and the two arcs CD and $C'D'$ are corresponding arcs; hence they are equal. But they are of opposite sense; or

FIG. 117.



$$CD = -C'D', \text{ or } CD = D'C'.$$

In the same manner we have

$$CD' = -C'D, \text{ or } CD' = DC',$$

where CD' and DC' are to be taken in the same sense.

This is also true if D and D' coincide, so that the secant DD' becomes the tangent at A parallel to the secant CC'; hence

$$CA = AC', \text{ or } CA = -C'A.$$

This may be expressed thus :

Theorem : 1. *Two parallel chords cut a circle in points such that the arcs joining one end point of the first to either end point of the second equals the arc joining the remaining end points in the opposite sense.*

If CC' and DD' are parallel, then

$$CD = D'C' \textit{ and } CD' = DC'.$$

2. *The arcs between the point of contact of a tangent and the end points of a chord parallel to it are equal and opposite; or*

If CC' is a chord parallel to the tangent at A, then

$$CA = AC'.$$

§ 239. The theorems converse to these also hold.

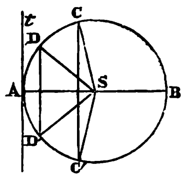
Converse Theorem: 1. *If two arcs of a circle are equal, then the chords which join the initial point of either to the end point of the other are parallel. In symbols*

If the arcs CD and $D'C'$ are equal, then the chords CC' and DD' are parallel.

2. *The tangent at the mid point of an arc is parallel to the chord joining the end points of the arc.*

For if arc CD=arc $D'C'$, then the diameter AB,

FIG. 118.



which bisects the angle CSC', bisects also the angle DSD', because the angles CSD and $D'SC'$ are equal, standing upon equal arcs. Hence C, C' and D, D' are pairs of corresponding points; their joins CC' and DD' are therefore parallel.

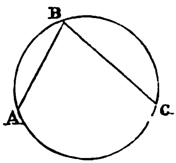
Further, the diameter SA which bisects the arc CC' bisects also the angle CSC', so that C and C' are symmetrical with regard to that diameter, and their join is therefore perpendicular to it, hence parallel to the tangent at its end point A.

§ 240. **Definition**: *An angle, which has its vertex on the circle and has for limits two chords of the circle, is called an* ANGLE AT THE CIRCUMFERENCE, *subtended by the arc joining the other end points of the chords.*

Thus in fig. 119 the angle ABC is subtended *by*, or stands upon, the arc AC, both the angle and

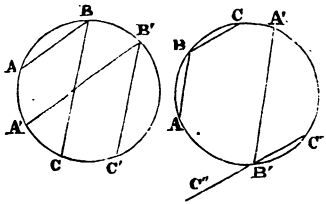
the arc being taken in the same sense, so that if a line through B describes the angle $A B C$, its intersection with the circle describes the arc $A C$.

FIG. 119.



§ 241. If we take in the same circle two angles at the circumference with their limits parallel, these will be equal if they have the same sense, as in the first figure, but supplementary if they are of opposite sense, as in the second figure. In the latter case the angle $A B C$ = angle $A'B'C''$, where $B'C''$ is the continuation of $C'B'$.

FIG. 120.



If the angles are of the same sense, and therefore equal, they stand upon arcs $A C$ and $A' C'$, which are of the same sense. But as $A B$ is parallel to $A'B'$, and $B C$ parallel to $B' C'$, arc $A A' =$ arc $C C'$, for they are both equal to arc $B B'$. Hence arc $A C =$ arc $A' C'$; or *the angles $A B C$ and $A'B' C'$ stand upon equal arcs.*

If the two angles are of opposite sense, $A B$ is parallel to $B'A'$ and $B C$ parallel to $B' C'$, whence, as before, $A A'$ equals $C C'$.

But $A C = A A' + A' C$ and $A' C' = A' C + C C'$.

From this it follows, as before, since $A A' = C C'$, that $A' C' = A C$. But angle $A B C$ stands on arc $A C$, and angle $A'B' C'$ stands on an arc equal to a *circumference* – arc $A' C'$ or on arc $(-A' C')$.

In this case, therefore, the angles $A B C$ and $A' B' C'$

stand upon arcs which complete the circumference. Such arcs are therefore said to be *supplementary*. Hence we have

Theorem: *Two angles at the circumference which have their limits parallel stand upon equal arcs if they are equal, but they stand upon supplementary arcs if they are supplementary.*

§ 242. The converse also holds, as follows:

Theorem: *If two angles at the circumference stand upon equal arcs, and one limit of the one is parallel to one limit of the other, then the second limit to the one is parallel to the second limit of the other and the angles are equal.*

For through the vertex of the second angle only one line can be drawn which is parallel to the second limit of the first angle, and one line only which, together with the first limit of the second angle, makes an angle equal to the first.

§ 243. If we now consider two angles at the circumference standing upon the same arc or upon equal arcs, we can slide the one along the circle till its first limit is parallel to the first limit of the second angle; the other limits will then be parallel and the the angles equal. Hence

Theorem: *Two angles at the circumference standing upon the same arc, or upon equal arcs, are equal.*

Similarly it follows that

Theorem: *Two angles at the circumference which stand upon supplementary arcs are supplementary.*

And as a special case of the last

Corollary: *Two angles at the circumference, whose limits meet the circumference on the same two points, but*

which have their vertices on opposite sides of the chord joining these points, are supplementary.

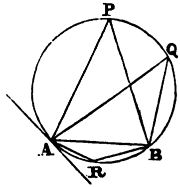
§ 244. Considering a tangent as the limiting case of a secant cutting in two coincident points, we get the following theorem—

Theorem : *The angle between the tangent to a circle and a chord which passes through the point of contact is equal to the angle at the circumference standing upon that chord and having its vertex on the side of it opposite to that on which the first angle lies.*

§ 245. These theorems may also be stated thus:—

Theorem : *The lines which join any point on a circle to two fixed points on it intersect under constant angles.*

FIG. 121.



§ 246. **Theorem** : *Equal arcs, or equal angles at the centre, or equal angles at the circumference, or supplementary angles at the circumference, are subtended by equal chords.*

And conversely

Theorem : *Equal chords subtend (1st) equal or supplementary arcs; (2nd) equal angles at the centre; and (3rd) equal or supplementary angles at the circumference.*

These theorems are proved like those in § 230 by sliding the one arc along the circumference till it coincides with the other arc which is equal to it.

§ 247. In the same manner it is proved that

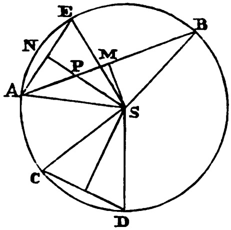
Theorem : *Equal chords are equidistant from the centre.*

And conversely

Chords which are equidistant from the centre are equal.

§ 248. If AB and CD are two unequal chords, $AB > CD$, the triangles ASB and CSD, where S denotes the centre of the circle, have the sides SA, SB equal to the sides SC, SD, and of the third sides $AB > CD$; therefore (§ 194)

FIG. 122.



$$\angle ASB > \angle CSD.$$

Of two unequal chords the greater subtends the greater angle at the centre.

The converse is proved in the same manner. If now the triangle CSD be turned about S till SD coincides with SA, then SC will fall within the angle ASB at SE, say. The mid point N of the chord AE and the centre S lie thus on opposite sides of AB. The line SN, that is, the perpendicular to AE, will therefore cut AB at some point P. It follows that $NS > PS$, whilst $PS > MS$, if M is the mid point of AB. Hence the distance NS of the smaller chord AE or CD is greater than the distance SN of the greater chord AB. Hence

Theorem : *Of two unequal chords that is the greater which is nearer the centre.* And conversely

Of two unequal chords the greater is nearer the centre.

Corollary : *A diameter is greater than any other chord*, its distance from the centre being the smallest *possible.*

EXERCISES.

(1) If from points on the circumference of a circle segments of lines be drawn equal, parallel, and of the same sense as a given segment drawn from the centre, what will be the *locus* of their extremities?

(2) Determine the *locus* of the mid point of a line connecting a fixed point with a point on the circumference of a given circle.

(3) One side of a triangle being given, as well as the length of the median line drawn from one of its extremities, find the *locus* of the vertex opposite the given side.

(4) A segment of constant length slides with its ends along the limits of a right angle. Find the locus of its mid point. (The carpenter's 'trammel' is an instrument for drawing this locus.)

(5) The tangents at the extremities of any diameter of a circle are parallel to one another; and conversely, the straight line which connects the points of contact of two parallel tangents is a diameter.

(6) Each of the angles at the base of the isosceles triangle formed by any chord of a circle and the tangents at its extremities is equal to an angle at the circumference subtended by the arc within the triangle.

(7) Every trapezium inscribed in a circle is symmetrical. (A trapezium is a quadrilateral with two parallel sides.)

(8) The end points of two equal chords, and likewise those of two parallel chords, of a circle are the vertices of a symmetrical trapezium.

(9) If through any point within or without a circle two lines are drawn cutting the circle, then any one of the four angles contained by them will be equal in magnitude and sense to an angle at the circumference subtended by an arc equal to the sum of the arcs which are intercepted by the limits of that angle produced, if necessary, beyond the vertex. (How ought this theorem to be stated if no account be taken of the sense of the angles?)

(10) What corollaries are deducible from the last proposition if the two lines are at right angles to one another?

(11) Of all chords which pass through a given point within a circle, which is the shortest?

(12) The length of a chord being given, determine the *locus* of its mid point.

DRAWING.

A circle on paper is generally drawn by aid of a pair of compasses with one point formed by a pencil or a pen. (Take care not to make a big hole with the point inserted at the centre.) For drawing large circles joiners and carpenters often use the 'trammel' [Ex. (4)].

(13) Draw a circle having its centre at a given point and its radius of a given length (take radius equal to, say, $\frac{1}{2}$, 1, and 2 inches).

(14) At a given point on a circle draw a tangent.

(15) From a point without a circle draw a tangent to a circle. (This is done by placing the straight-edge through the point and touching the circle.)

(16) A tangent being drawn to a circle, find the point of contact. (Draw perpendicular from centre.)

(17) Draw a circle which shall have a given point as centre and touch a given line.

(18) By aid of set squares divide the circumference of a circle into two, three, four, six, eight, and twelve equal parts.

(19) In a given circle inscribe regular convex polygons of 3, 4, 6, 8, and 12 sides. (A polygon is said to be inscribed in a circle if its vertices lie on the circle.)

(20) In a given circle inscribe regular star-polygons of 6, 8, and 12 sides.

(21) Bisect an arc of a circle (by perpendicular from centre to chord).

(22) Divide the circumference of a circle into 16 equal parts, and draw (in separate figures) all possible regular star-polygons which have the points of division as vertices.

(23) Divide the circumference of a circle into 24 equal parts, and draw (in separate figures) all possible regular star-polygons having their vertices at the points of division. How many of these star-polygons are polygons proper, and how many are *combinations* of regular triangles, squares, &c.?

CHAPTER XIII.

THE CIRCLE AS ENVELOPE.

§ 249. UP to this we have only considered the circle as described by a moving point. We shall now see that a circle may also be generated by a moving line.

We need only remember that a tangent has its distance from the centre equal to the radius, and that all lines which have their distances from the centre equal to the radius are tangents, in order to see the truth of the following theorem—

Theorem : *The set of lines equidistant from a fixed point consists of the tangents to a circle which has the fixed point for its centre and the distance of the lines from it for its radius.*

§ 250. If all lines in a set are tangents to a curve, it is said that the lines *envelope* that curve, or the curve is said to be the *envelope* of the set. The last theorem thus becomes—

Theorem : *The lines equidistant from a fixed point envelope a circle which has the fixed point for its centre and the constant distance for its radius.*

§ 251. We have obtained thus a second mode of generating curves. The first was to move a point so as to describe a curve ; the other is to move a line so as to envelope a curve. The first method is constantly used in drawing curves by means of a pen or pencil ; the second method is not quite so familiar.

But fig. 123 will show that a curve may be clearly determined by drawing a number of lines according to some law. But we may also draw a curve directly by a continuous motion of a line. If we take a plane board, and cover it evenly with a thin layer of some dry powder, such as sand or fine sawdust, and then move a straight-edge in any manner, always keeping its edge on the board, it will remove the sand from those parts of the board swept over by the moving edge, and push it together at other parts.

Fig. 123.

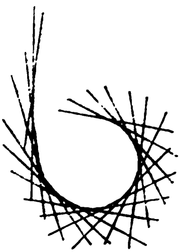
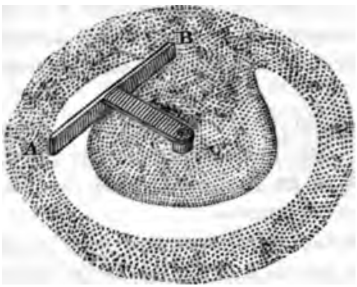


Fig. 124.



These will appear bounded by a curve, and this will be the envelope of the moving line.

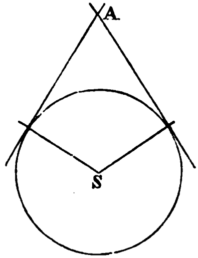
For instance, if a ruler AB be fixed, as in fig. 124, to a centre, it will thus trace out a circle, of which, in the figure, the greater part is supposed drawn.

§ 252. We know (§ 216) that through one point two lines can be drawn which have the same distance from a given point, provided that that distance is less than the distance between the points.

Hence if S is the centre of a circle, and A a point *without a circle*—that is, at a distance from the centre

greater than the radius—then two lines may be drawn from A which have their distance from S equal to the radius, and which are therefore tangents to the circle. But if A is on the circle only one such line may be drawn, and if A is within the circle, none. Hence

FIG. 125.

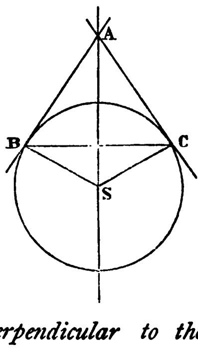


Theorem : *To a circle two tangents may be drawn from a point without it, one from a point on it, and none from a point within it.*

§ 253. The two lines equidistant from S which can be drawn through a point A are symmetrical with regard to the line SA (§ 216), so are the perpendiculars from S upon them. Hence

Theorem : *If two tangents to a circle be drawn from a point A without the circle, then these are symmetrical to the line AS joining A to the centre of the circle. These tangents, together with the radii drawn to the points of contact, form a kite.*

FIG. 126.



Corollary I. : *The line SA bisects*

1st. *The angle between the two tangents.*

2nd. *The angle between the radii drawn to the points of contact.*

3rd. *The chord joining the points of contact. This is called the* CHORD OF CONTACT, *and is perpendicular to the line AS.*

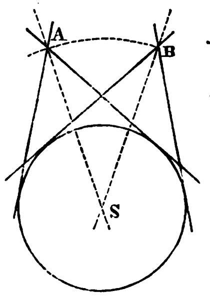
4th. *The arcs into which the points of contact divide the circumference.*

Corollary II.: *The diameter perpendicular to a chord passes through the intersection of the tangents drawn at the end points of the chord.* For there is but one perpendicular bisector of the chord, and but one line joining the centre to the join of the two tangents.

Corollary III.: *Of the two tangents drawn from a point without a circle the segments between the common point and the points of contact are equal in length and equally inclined to the chord of contact.*

§ 254. **Theorem** : *If A and B are two points without a circle and equidistant from the centre, then the tangents drawn from A to the circle are equal to those drawn from B, and include equal angles; their chords of contact, and also the arcs subtended by these chords, are equal.*

FIG. 127.



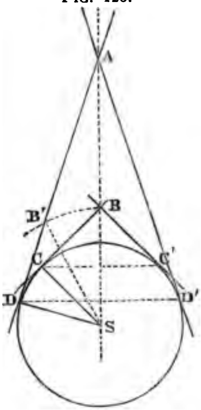
For we may turn the circle, together with the tangents drawn from A, round the centre till A coincides with B, in which case also the tangents will coincide (fig. 127).

§ 255. **Theorem** : *If A and B* (fig. 128) *are two points without the circle on the same diameter, then the tangents from A and B will form a kite, with the diameter A B S as axis of symmetry.*

The chords of contact C C′ and D D′ are parallel.

If A is at a greater distance from S than B, if $AS > BS$, then the tangents from A are greater than those from B, but they include a smaller angle.

FIG. 128.



For if we turn B, together with its tangents, about the centre S till B comes to B', a point on the tangent from A, then $SB' < SA$. We have now from S two obliques SB' and SA, of which the first is the smaller. Its departure is therefore the smaller—that is, if D is the point of contact of the tangent from A, $DB' < DA$. But $DB' = CB$, so that $BC < AD$.

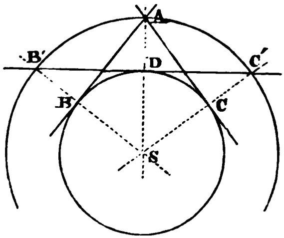
The last part of the theorem may be stated more generally. If we turn B with the tangents from it to any other position, the length of a tangent drawn from it to the circle remains unaltered.

§ 256. When a circle is given with centre at S and a point A without it, it is easy to find, by construction, the points of contact of the tangents drawn from A to the circle. These being known, the tangents are of course found at once, by joining the points of contact to A.

To find the points of contact, let us draw a circle with centre S and radius SA. Then all points on this circle have their tangents to the given circle equal in length. If the radius SA cut the given circle at D, and if the tangent at D cut the new circle at B'

and C', then the chord $B'C'$ is perpendicular to the radius. The radius therefore bisects the chord, and also the arc $B'C'$, so that arc $B'A=$ arc AC'. If we now turn the triangle $B'C'S$ about S, then the chord $B'C'$ remains always at the distance SD from S, and remains therefore a tangent to the given circle. If we turn till C' comes to A, then A will come to B' and D to B, where the line SB' cuts the given circle. The line $C'D$ takes, therefore, the position AB, touching at B. Hence B is the point of contact of a tangent from A to the circle. The point C where the line SC' cuts the given circle is, for the same reason, the other point of contact. Hence we have the following construction:

FIG. 129.



Let the line SA cut the given circle at D. Draw the tangent at D and let B', C' be the points where this tangent is cut by a circle having its centre at S and passing through A; then the points B and C where the lines SB' and SC' cut the given circle are the points oj contact of tangents drawn from A.

EXERCISES.

(1) Determine the envelope of equal chords of a circle.

(2) Through a fixed point a perpendicular is drawn to a *movable tangent* of a circle, and through the mid point of this

perpendicular a line parallel to the tangent. What is the envelope of this line?

(3) Two tangents to a circle intersect at constant angles. Find the envelope of the bisectors of those angles.

(4) The diameters of a circle which pass through the points where two fixed tangents are cut by a movable tangent include constant angles.

(5) Two parallels are cut by a common perpendicular in A and B, and by another line in P and Q, in such a manner that the perpendicular OR drawn from the mid point of AB to the line PQ divides the latter into two segments PR and RQ, which are equal respectively to AP and BQ. Find the envelope of PQ.

(6) A right angle turns about its vertex, which is fixed halfway between two parallels. What is the envelope of the line which joins the points where the limits of the right angle cut the parallels?

(7) An angle which is equal in magnitude to half the angle between the perpendiculars drawn from its vertex to two given lines turns about its vertex which is equidistant from the given lines. Find the envelope of the line which joins the points where the limits of the angle cut the given lines.

(8) A kite $ABCD$, with AB as axis, has the vertex A at a given point and the side CB in a given line. Determine the envelope of the side BD.

DRAWING.

(9) Draw figures to the exercises (1)–(5).

(10) Draw a sufficient number of the set of lines described in questions (6), (7), (8), to bring out the envelope.

(11) About given circles circumscribe regular convex polygons of 3, 4, 6, 8, and 12 sides. (A polygon is said to be circumscribed about a circle if each of its sides touches the circle.)

(12) About given circles circumscribe regular star-polygons of 6, 8, 12, and 16 sides.

(13) Draw a set of circles

1. With equal radii having their centres on a given line or

on a given circle. (They envelope two lines parallel to the given line or two circles concentric to the given circle.)

2. Having their centres on a given line and touching another given line. (They envelope a third line, passing through the join of the first two.)

3. Having their centres on a given circle and all passing through a fixed point within, on, or without the given circle. (They envelope a beautiful curve, which is, however, not of an elementary character.)

CHAPTER XIV.

CONDITIONS WHICH DETERMINE A CIRCLE.

§ 257. LET us now see how many conditions are necessary to determine a circle; in particular, how many points, and reciprocally how many tangents of a circle, may be assumed arbitrarily.

If it is required to draw a circle through a given point A, any point S may be taken as centre, and its distance SA from the point as radius.

If it is required to draw a circle touching a given line a, then any point S may be taken as centre, and its distance Sa from the line as radius.

If a circle has to be drawn through two given points A and B, the centre has to be equidistant from A and B; hence it must lie on the perpendicular bisector of AB (§ 208).

If a circle has to be drawn touching two given lines a and b, the centre has to be equidistant from a and b; hence it must lie on one of the bisectors of the angles ab (§ 208).

Theorem a: *An infinite number of circles may be drawn through two given points; the locus of their centres is the perpendicular bisector of the segment joining the two given points.*

Theorem b: *An infinite number of circles may be drawn which touch two given lines; the locus of their centres consists of the bisectors of the angles between the given lines.*

§ 258. A circle, then, is not determined by two points or by two tangents. But

if three points A, B, C are given, we know (§ 210) there is always one, and only one, point S equidistant from A, B, C, provided that these points do not lie in a line. The circle, having this point S as centre, and $SA = SB = SC$ as radius, will pass through A, B, and C, and this will be the only circle through them. It is said to be *circumscribed* about the triangle ABC.

if three lines a, b, c are given, we know (§ 212) there are always four points S, S_1, S_2, S_3 equidistant from the three lines a, b, and c, provided that the three lines form a triangle

Each of these is the centre of a circle touching all three lines. One has its centre within the triangle abc, and is said to be *inscribed* in it; the others are said to be *escribed*.

We thus see that *three points determine one circle*, whilst by *three tangents four circles are determined.* The first case is the more important one. It gives the following theorem:

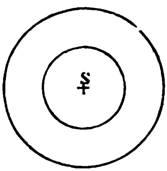
Theorem: *Through three points not in a line, one, and only one, circle can be drawn*; or, what is the same thing

Two circles which have three points in common coincide.

§ 259. The contra-positive form of the last proposition gives

Theorem : *Two different circles cannot have more than two points in common.*

FIG. 130.



Two circles, however, need not have any point in common. For instance, if we call two circles which have the same centre *concentric* (fig. 130), we see at once that

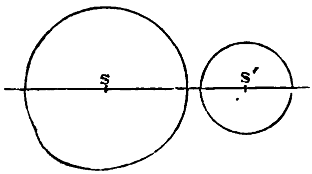
Two concentric circles with unequal radii have no point in common; the one which has the smaller radius lies altogether within the other.

If two circles have two points in common they are said to *cut*, or to intersect, and the line joining the common points is called the *common chord.*

It may also happen, as we shall presently see, that two circles have only one point in common. In this case the two circles are said *touch*, and the common point is called the *point of contact.*

§ 260. To investigate these different cases let us suppose any two circles given, with centres at different

FIG. 131.



points S and S'. The line $S\,S'$, called *the line of centres*, contains a diameter of each, and is therefore a common axis of symmetry.

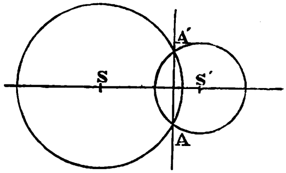
Theorem: *Two circles have always a common axis of symmetry, and in general only one, viz. their line of centres. If they have two*

common axes of symmetry, and are unequal, they are concentric and have an infinite number of such axes. For in this case the line of centres becomes indeterminate.

§ 261. From this there follows that

If two circles have one point A in common which is not on the line of centres, then they have necessarily a second point A' in common, viz. the point which is, with regard to the line of centres, symmetrical to A.

FIG. 132.

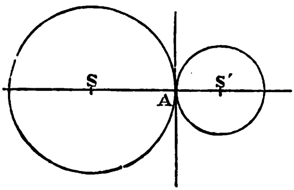


If two circles have two points in common, these must be corresponding points with regard to the line of centres, for there are but two points of intersection possible, and to a point of intersection corresponds a point of intersection. Hence

Theorem: *The common chord of two intersecting circles is bisected perpendicularly by the line of centres.*

§ 262. **Contact of Circles.**—If, however, the two circles have a point A in common, which lies on the line of centres, then they cannot have any other point in common, as the point A corresponds to itself; and conversely, if two circles have but one point in common, this point must lie on the line of centres. Or

FIG. 133.



Theorem: *If two circles touch, the point of contact lies on the line of centres, and the line perpendicular to the latter through the point of contact is a common tangent.*

§ 263. **Different Positions of two Circles.**—If two circles are given they may have one of the following positions:

If the two circles have two different points A and B in common, the two circles intersect. A point which moves along one of the circles, crosses the other circle on passing through A, and recrosses on passing through B. Hence if the moving point is first without the second circle, it will lie within it when it has passed A. It will remain within it till it comes to B, and will be outside after having passed B. Two circles, therefore, which intersect, lie each partly within and partly without the other. If, however, two circles touch, then a point on moving along the one circle will never cross the other circle; hence it will always remain within or without the second circle, according as it was at first within or without it.

Of two circles which touch, the one lies either altogether within the other, or each of them lies altogether without the other.

The circles are said to have in the first case *internal*, in the second *external*, *contact*. If two circles have no point in common, one lies either totally within the other or each lies without the other.

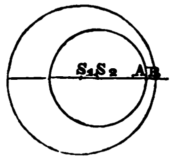
§ 264. These relations will become still clearer from the following considerations, which at the same time will enable us to state the conditions which must hold in each case between the radii of the circles and the distance between the centres.

Let S_1 and S_2 be the centres of the circles; let r_1 and r_2 denote their radii, r_2 being supposed the smaller; and let d denote the distance between the centres.

We shall at first suppose that the circles are concentric, and then that the centre S_2 of the smaller circle moves along a line to a greater and greater distance from S_1.

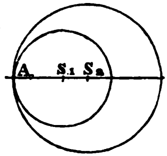
If the circles are concentric the smaller one, with radius r_2, will lie altogether within the other. If we next move S_2 to the position in fig. 134, we see that $S_1 S_2$ together with $S_2 A$ is less than $S_1 B$, or that $S_1 B - S_2 A > S_1 S_2$, or $r_1 - r_2 > d$; hence also $r_1 + r_2 > d$. This goes on till A coincides with B (fig. 135): in this case the circles touch internally. Now $S_1 S_2 + S_2 A = S_1 B$, or $r_1 - r_2 = d$; hence $r_1 + r_2 > d$.

FIG. 134.



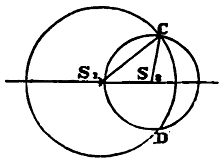
If S_2 moves further, the two circles will cut at two points C and D. $S_1 C S_2$ is a triangle (fig. 136). Hence the difference of two sides $S_1 C$ and $S_2 C$ is less, and their sum greater than the third side $S_1 S_2$; or $r_1 - r_2 < d$, $r_1 + r_2 > d$.

FIG. 135.



Moving S_2 still further, the circles come to a position of external contact. In this case (fig. 137) the distance between the centres is equal to the sum of the two radii, or $r_1 + r_2 = d$. Hence also $r_1 - r_2 < d$.

FIG. 136.



On increasing d again, the circles will separate and have no point in common (fig. 138). Now the distance d is greater than the

sum of the two radii. Hence $r_1+r_2<d$. We may therefore classify the different possible cases thus :

I. *One circle lies within the other,* $r_1-r_2>d$.
II. *Internal contact,* $r_1-r_2=d$.
III. *Intersection,* $r_1-r_2<d$; $r_1+r_2>d$.
IV. *External contact,* $r_1+r_2=d$.
V. *One circle lies entirely without the other,* $r_1+r_2<d$.

FIG. 137.

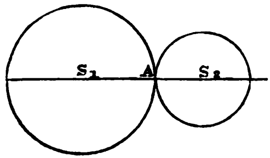
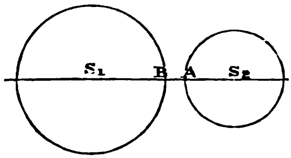


FIG. 138.



§ 265. If now two circles are given, then one of the above five positions must hold, and only one can hold. Of the above five relations also one, and only one, must hold. For the difference r_1-r_2 must be less than, equal to, or greater than d; and likewise the sum r_1+r_2 must be either less than, equal to, or greater than d.

Combining these results, we get the above five cases, and these only. For in the first two cases the difference between the radii is greater than, or equal to, the distance between the centres ; hence the sum of the radii is necessarily greater than that distance. Similarly in the last two cases the sum of the radii is equal to or less than the distance between the centres; the difference between the radii is therefore certainly less than that distance d.

From this it follows at once that *if we know the*

relations between the difference and the sum of the radii as compared to the distance between the centres, then we also know in which of the five possible positions the two centres lie.

If, for instance, we know that the sum of the radii equals the distance between the centres, then we know that the circles must have external contact, because this is the only case in which the relation in question holds.

The same reasoning applies to the other cases.

§ 266. **Common Tangents to two Circles.**—All these last results (from § 259 to § 265) depend upon the property that two circles passing through the same three points coincide. Theorems reciprocal to these do not exist, because three tangents determine four circles; hence two circles which touch the same three lines do not necessarily coincide.

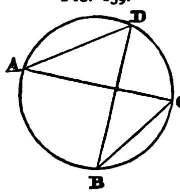
From this it follows that two circles may have at least three common tangents. But two circles have their line of centres as an axis of symmetry. To a common tangent corresponds, with regard to that axis, a common tangent to the two circles.

It follows, therefore, that the tangents common to two circles occur in pairs, corresponding ones intersecting on the axis of symmetry. Hence as our circles have three common tangents, they must in general have four tangents in common. This is the greatest number of tangents which two circles can have in common.

A fuller investigation of the tangents common to two circles may be postponed till after the investigation of similar figures.

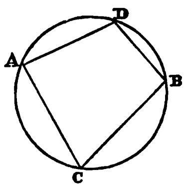
§ 267. **Four Points on a Circle.**—As three points determine a circle, it is in general impossible to draw a circle which passes through four given points A, B, C, D; the circle determined by the three points A, B, C will, as a rule, not pass through the point D. But it is possible that this may happen. In this case we have four points on a circle. Such points are said to be *concyclic*.

FIG. 139.



If we join two of these—say, A and B—to each of the others, we obtain two angles at the circumference ACB and ADB. These stand either on the same side (fig. 139) or on opposite sides of AB (fig. 140), and hence either on the same or on supplementary arcs of the circle. In the first case the angles are equal, in the second they are supplementary.

FIG. 140.



Theorem: *If four points are concyclic, then the angles included by the lines which join two of the points to each of the others are either equal or supplementary, according as they lie on the same or on opposite sides of the line joining the first two points.*

§ 268. **Quadrilaterals Circumscribed about a Circle.**—A quadrilateral whose sides are tangents to a circle is said to be circumscribed about a circle.

As three tangents determine a finite number, namely four circles, it is, as a rule, impossible to draw a circle which shall touch four given lines, or shall be *inscribed in* a quadrilateral. If a quadrilateral be *such that a* circle may be inscribed in it, it must have

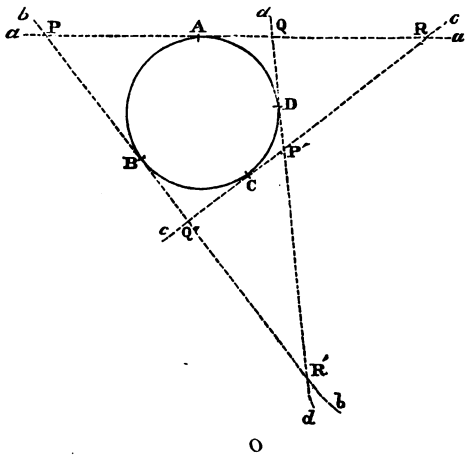
properties reciprocal to those considered in the last paragraph. To bring out this reciprocity we will state the above theorem about a quadrilateral inscribed in a circle in a slightly different form, together with the reciprocal theorem, as follows :—

If a quadrilateral be inscribed in a circle, then either the sums, or else the differences of pairs of opposite angles, are equal, according as the quadrilateral is convex or not.

If a quadrilateral be circumscribed about a circle, then either the sums or else the differences of opposite sides are equal.

The sums are equal if the quadrilateral is convex, or has a re-entrant angle, but the differences are equal if two opposite sides intersect.

FIG. 141.



In fig. 141 a, b, c, d are four tangents to a circle; they determine three different quadrilaterals according as b or c or d be taken as the side opposite a.

In the quadrilaterals $PQP'Q'$ and $PRP'R'$ it is the sum of opposite sides which is constant, whilst in the quadrilateral $QRQ'R'$, of which the opposite sides, QR', $Q'R$, intersect at P', it is the difference.

The proof of this theorem follows at once from the fact that the tangents drawn from a point to a circle are equal in length (§ 253).

To take the first quadrilateral, we have (fig. 141)

$$PA = PB,\ QA = QD,\ Q'C = Q'B,\ P'C = P'D.$$

Hence $PA + AQ + P'C + CQ' = PB + BQ' + P'D + DQ$;

or, $$PQ + P'Q' = PQ' + P'Q.$$

And this is the theorem. The other cases are proved in the same way.

EXERCISES.

(1) If a point has equal distances from more than two points on a circle, it is the centre of that circle.

(2) State the theorem contra-positive to the last, and discuss the question whether two equal segments may be drawn from any point to points on a circle.

(3) Two intersecting circles make equal angles at the two points of intersection.

(By the angle between two intersecting circles is meant the angle between the tangents to the circles at the point of intersection. The circles are said to cut *orthogonally* when this angle is a right angle.)

(4) What is the magnitude of the angle made by two circles each of which passes through the centre of the other?

(5) If two circles cut one another orthogonally, the two

radii to each point of intersection will be perpendicular to one another. Prove also the converse of this, and examine in which case the common chord will be equal to the segment between the two centres.

(6) A circle may be circumscribed about every symmetrical trapezium and inscribed in every kite.

(7) In which case may a kite be inscribed in, or a symmetrical trapezium be circumscribed about, a circle?

(8) A parallelogram inscribed in a circle is a rectangle.

(9) A parallelogram circumscribed about a circle is a rhombus.

(10) Two equal circles have the mid point between their centres a centre of symmetry; they have also two axes of symmetry. If the circles touch, the point of contact is the centre of symmetry.

(11) How has a line through one of the common points of two intersecting circles to be drawn in order that the two circles may intercept equal chords on it?

(12) Through one of the points of intersection of two circles draw the line on which the two circles intercept the greatest possible segment.

(13) The centres of two circles, whose radii are five and six inches respectively, are four inches apart. Will the circles intersect or not? How far apart must their centres be placed in order that the circles may touch?

(14) If any two lines be drawn through the point of contact of two circles, the joins of their second intersections with each circle will be parallel to one another.

DRAWING.

By the aid of a pair of compasses perform the following five constructions:

(15) Draw a triangle which has its sides of given length.

(16) On a given base construct an isosceles triangle.

(17) Bisect a given segment. (Construct a kite having the segment as its transverse axis.)

(18) Bisect a given angle. (Construct a kite having the given *angle at one* end of the axis.)

(19) Through a given point on, or off, a given line draw a perpendicular to that line.

(20) Draw a circle

1. Through three given points (circumscribed about a triangle).

2. Inscribed in a triangle.

3. Having a given radius and touching two given lines.

4. Having a given radius passing through a given point and touching a given line or a given circle.

5. Touching a given circle at a given point and passing through a given point.

(21) Draw figures to the above exercises, from (3) to (14) inclusive.

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